List of Knowledge Points

Algebra

Negative Exponents and Fractional Exponents

For any number a such that $a \neq 0$, we have $a^0 = 1$, and $a^{-n} = \frac{1}{a^n}(a \neq 0)$.

Note: 0^0 is not defined

When you have a fractional exponent, the numerator is the power and the denominator is the root. In the variable example $x^{\frac{a}{b}}$, where a and b are positive real numbers (a and b will only be integers in our current study) and x is a real number, a is the power and b is the root, namely

$$x^{\frac{a}{b}} = \sqrt[b]{x^a}$$

Polynomial Formulae

1. Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$

2. Perfect Square Formula: $(a \pm b)^2 = a^2 \pm 2ab + b^2$

3. Two important formulae to remember and understand with:

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$a^2 + b^2 + c^2 + ab + bc + ca = \frac{1}{2}\left[(a+b)^2 + (b+c)^2 + (c+a)^2\right]$

4. Sum of Cubes Formula: $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$

5. Difference of Cubes Formula: $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$

6. Perfect Cube Formulae:

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - c^3$

Remainder Theorem and Factor Theorem

The polynomials with respect to x can be simply represented by $f(x)$ or $g(x)$. Here f is a series of operations on x. $f(a)$ means the value of $f(x)$ when $x = a$. For example, if we use $f(x)$ to represent $2x^2 + 5x - 6$, then $f(x) = 2x^2 + 5x - 6$. $f(3)$ is the value of $2x^2 + 5x - 6$ when $x = 3$, i.e., $f(3) = 2 \times 3^2 + 5 \times 3 - 6 = 27$.

The quotient of the division of $f(x)$ divided by $g(x)$ is $q(x)$, and the remainder is $r(x)$.
$f(x) \div g(x) = q(x)\ R\ r(x)$, i.e., $f(x) = g(x)q(x) + r(x)$.
Especially, when the divisor $g(x)$ is a linear term $(x-a)$, the remainder $r(x)$ must be a constant. In this case, the remainder can also be written as r, therefore $f(x) = (x-a) \cdot q(x) + r$.

Polynomial **Remainder Theorem** states that the remainder of the division of a polynomial $f(x)$ by a linear polynomial $x - a$ is equal to $f(a)$.
In particular, $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$, and this is the **Factor Theorem**.

We can see that the factor theorem is just a special case of the remainder theorem when $r = 0$.

Rational Root Test

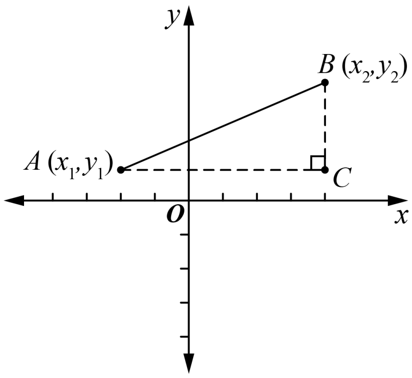
Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, (a_n \neq 0)$ be a polynomial of degree n with respect to x:
Property ①: If the coefficient of the first term $a_n = 1$, and the polynomial contains the factor $(x-q)$ (q is an integer), then q must be the factor of the constant a_0.
Property ②: If the coefficient of the first term $a_n \neq 1$, and the polynomial contains the factor $px - q$, where p, q are coprime integers, then p must be the factor of the coefficient of the first term a_n, and q must be a factor of the constant a_0.

Cartesian Coordinate System

Basics in the coordinate plane:

By utilizing the **Cartesian coordinate system**, we can express a geometric shape algebraically and also derive the corresponding geometric representation for an algebraic expression. This enables us to transform geometric problems into algebraic problems, or vice versa, facilitating subsequent problem-solving.

1. The Distance Formula between Two Points: The distance between point $A\left(x_1, y_1\right)$ and point $B\left(x_2, y_2\right)$ is $\sqrt{\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2}$.



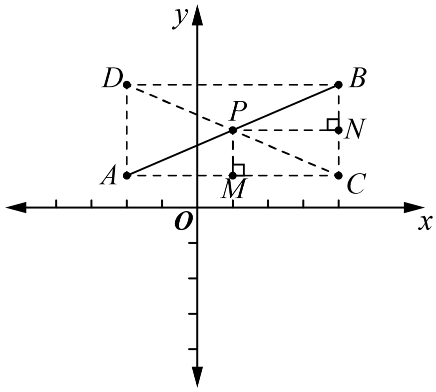
In the right triangle ABC, according to the Pythagorean Theorem, $AB^2=AC^2+BC^2$.

Thus $AB=\sqrt{AC^2+BC^2}$, $AC=\left|x_1-x_2\right|$ and $BC=\left|y_1-y_2\right|$.

Therefore, $AB=\sqrt{\left(\left|x_1-x_2\right|\right)^2+\left(\left|y_1-y_2\right|\right)^2}=\sqrt{\left(x_1-x_2\right)^2+\left(y_1-y_2\right)^2}$.

2. Midpoint Formula

The coordinates of midpoint P between point $A\left(x_1, y_1\right)$ and point $B\left(x_2, y_2\right)$ are $P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.



The x-coordinate of point P is $AM - x_A = \frac{1}{2}AC - x_A = \frac{x_2 - x_1}{2} - (-x_1) = \frac{x_1 + x_2}{2}$;

The y-coordinate of point P is $CN + y_A = \frac{1}{2}BC + y_A = \frac{y_2 - y_1}{2} + y_1 = \frac{y_1 + y_2}{2}$.

3. The distance between the point $P(X_0, Y_0)$ and the line with the equation $Ax + By + C = 0$ is denoted as $d = \left|\frac{AX_0 + BY_0 + C}{\sqrt{A^2 + B^2}}\right|$.

Two lines are perpendicular if and only if the product of their slopes is -1, or they are parallel to the x and y-axes respectively. It's important to note that lines parallel to the y-axis cannot be represented as $y = mx + b$. Consequently, in this chapter, we exclusively employ the general form of a line to express them.

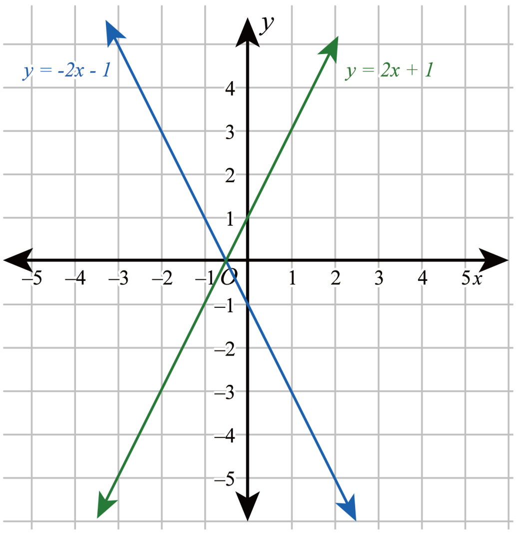
Functions

Linear Functions:

The fundamental idea behind transforming a function's graph is indeed to counteract the effects by modifying the function's expression. Therefore, for certain specific lines, we can calculate this using specialized formulas.

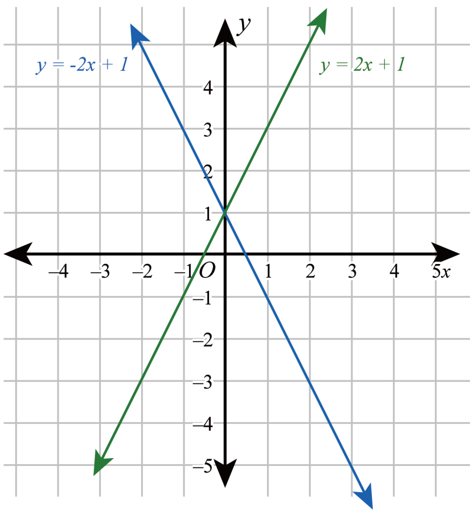
1. Symmetry about the x-axis:

The equation representing the line symmetric to the linear equation $Ax + By = C$ with respect to the x-axis is given by $Ax - By = C$.



2. Symmetry about the y-axis:

The equation representing the line symmetric to the linear equation $Ax + By = C$ with respect to the y-axis is $-Ax + By = C$.



3. Symmetry about the line $y = x$:

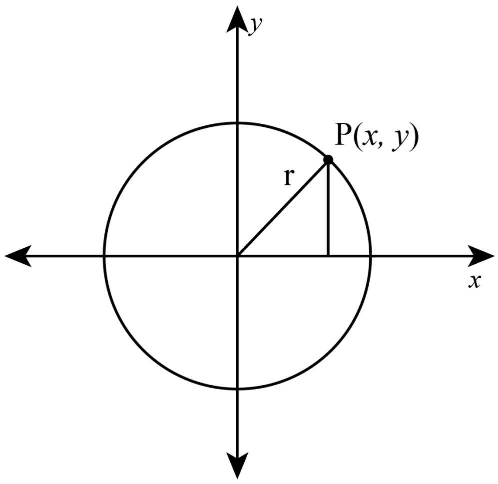
The equation representing the line symmetric to the linear equation $Ax + By = C$ with respect to the line $y = x$ is $Bx + Ay = C$.

4. Symmetry about the line $y = -x$:

The equation representing the line symmetric to the linear equation $Ax + By = C$ with respect to the line $y = -x$ is $Bx + Ay = -C$.

Circles:

A **circle** is the set of all points in a plane that are equidistant from a given point in that plane. The given point is called the **center** of the circle and the distance bewteen the center and any point on the circle is the **radius**.



The standard form of the equation of a circle with center at $(0,0)$ and radius r is:

$x^2+y^2=r^2$.

The standard form of the equation of a circle with center at (h,k) and radius r is:

$(x-h)^2+(y-k)^2=r^2$.

The general form of the equation of a circle is given as $x^2+y^2+Dx+Ey+F=0$.

Quadratic Functions:

Consider the parabola $y=ax^2+bx+c$ or $y=a(x-h)^2+k$ with $a\neq 0$.

- If $a>0$, the parabola opens upward; if $a<0$, the parabola opens downward.
- The axis of symmetry is $x=-\dfrac{b}{2a}$ or $x=h$.
- The vertex is at $\left(-\dfrac{b}{2a},\dfrac{4ac-b^2}{4a}\right)$ or (h,k).
- Extreme values:

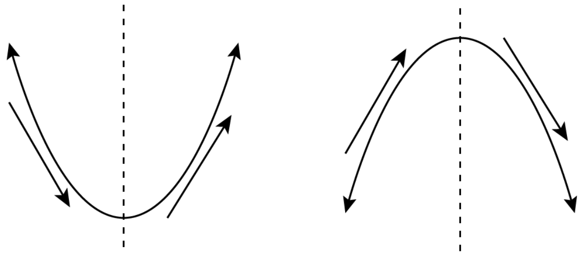


Figure 1 Figure 2

If $a > 0$, the minimum value of the quadratic function is $\frac{4ac - b^2}{4a}$ or k, as illustrated in Figure 1.

If $a < 0$, the maximum value of the quadratic function is $\frac{4ac - b^2}{4a}$ or k, as illustrated in Figure 2.

- Consider the increasing and decreasing intervals of $f(x) = ax^2 + bx + c$ with $a \neq 0$.

  As shown in Figure 1, when $a > 0$, $f(x)$ is decreasing to the left of the axis of symmetry and increasing to the right of the axis of symmetry.

  As shown in Figure 2, when $a < 0$, $f(x)$ is increasing to the left of the axis of symmetry and decreasing to the right of the axis of symmetry.

- The y-intercept is $(0, c)$ and the x-intercepts are all $(x, 0)$ where x is a solution to $ax^2 + bx + c = 0$.

Vieta's Formulas

In the context of any quadratic equation expressed as $ax^2 + bx + c = 0$, where $a \neq 0$, Vieta's formulas state that:

- The **sum of the roots** is given by $-\frac{b}{a}$.

- The **product of the roots** is given by $\frac{c}{a}$.

When working with integer coefficients a, b, and c, the value of the discriminant $\Delta = b^2 - 4ac$ must meet the following condition:

Discriminant	Nature of Solutions
$\Delta > 0$ and Δ is a perfect square	Two real, rational solutions
$\Delta > 0$ and Δ is not a perfect square	Two real, irrational solutions
$\Delta = 0$	One real, rational solution (a double root)

Vieta's Formulas can also be applied to higher degree equations.

For a cubic equation $ax^3 + bx^2 + cx + d = 0 (a \neq 0)$, suppose that the three roots are x_1, x_2, and x_3, respectively. Then we have:

$$\begin{cases} x_1+x_2+x_3=-\frac{b}{a} \\ x_1x_2+x_1x_3+x_2x_3=\frac{c}{a} \\ x_1x_2x_3=-\frac{d}{a}. \end{cases}$$

Similar conclusions also apply for quartic (fourth-degree) or even higher degree equations:

For the equation $a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0=0$, we have

$$\begin{cases} x_1+x_2+\cdots+x_{n-1}+x_n=-\frac{a_{n-1}}{a_n} \\ (x_1x_2+x_1x_3+\cdots+x_1x_n)+(x_2x_3+x_2x_4+\cdots+x_2x_n)+\cdots+x_{n-1}x_n=\frac{a_{n-2}}{a_n} \\ \vdots \\ x_1x_2\ldots x_n=(-1)^n\frac{a_0}{a_n}. \end{cases}$$

Arithmetic Sequences

An **arithmetic sequence** is a sequence of numbers such that the difference between the consecutive terms is constant.

The common difference of an arithmetic sequence is often denoted by d.

The sequence (a_n) is an arithmetic sequence with a common difference of d if and only if for any integer $n\geqslant 2$, $a_n-a_{n-1}=d$.

If the initial term of an arithmetic sequence is a_1 and the common difference of successive members is d, then the n^{th} term of the sequence (a_n) is given by $a_n=a_1+(n-1)d$, and in general, $a_n=a_m+(n-m)d$. This formula is called the **General Term Formula of Arithmetic Sequence**.

Properties of **Arithmetic Sequence:**

① If (a_n) is an arithmetic sequence, and $k+l=m+n(k,l,m,n\in\mathbb{N}^*)$, then $a_k+a_l=a_m+a_n$;

② If (a_n) is an arithmetic sequence, then $2a_n=a_{n-1}+a_{n+1}$, and $a_1+a_n=a_2+a_{n-1}=a_3+a_{n-2}=\cdots$;

③ If d_1 and d_2 are the common differences of the arithmetic sequences (a_n) and (b_n), respectively, then (pa_n+qb_n) is an arithemetic sequence with the common difference pd_1+qd_2 ;

④ If (a_n) is an arithmetic sequence, then a_k, a_{k+m}, a_{k+2m}, $\cdots$, where $k, m \in \mathbb{N}^*$ will form an arithmetic sequence with the common difference md.

The n^{th} partial sum of an arithmetic sequence is the sum of the first n terms in the sequence, often denoted by S_n.

The n^{th} partial sum of the arithmetic sequence (a_n), with the common difference d, can be computed by $S_n = \dfrac{n(a_1 + a_n)}{2}$, or $S_n = na_1 + \dfrac{n(n-1)}{2}d$.

Arithmetic Sequence partial sums have the following properties:

① If S_n denote the n^{th} partial sum of an arithmetic sequence whose common difference is d, then $(S_m, S_{2m} - S_m, S_{3m} - S_{2m}, \cdots)$ is an arithmetic sequence, with the common difference $m^2 d$;

② Let (a_n) be an arithmetic sequence.

Since $a_n = \dfrac{a_1 + a_{2n-1}}{2}$, $S_{2n-1} = \dfrac{(2n-1)(a_1 + a_{2n-1})}{2} = (2n-1)\, a_n$.

Since $a_1 + a_{2n} = a_n + a_{n+1}$, $S_{2n} = n(a_n + a_{n+1})$;

③ If (a_n) is an arithmetic sequence, then $S_n = An^2 + Bn$ for some $A \neq 0$, and $\left(\dfrac{S_n}{n}\right)$ is also an arithmetic sequence.

Geometric Sequences

A **geometric sequence** is a sequence of non-zero numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the **common ratio**.

The non-zero common ratio is often denoted by r.

The general form of a geometric sequence is $a, ar, ar^2, ar^3, ar^4, \ldots$, where $r \neq 0$ is the common ratio, and $a \neq 0$ is the sequence's initial value.

If the initial term of a geometric sequence is a_1 and the common ratio of successive members is r, then the n^{th} term of the sequence (a_n) is given by $a_n = a_1 r^{n-1}$, and in general, $a_n = a_m r^{n-m}$. This formula is called the **General Term Formula of Geometric Sequence**.

Let (a_n) be a geometric sequence with a common ratio of $r \neq 0$.

① If $p+q=m+n$ for some $m,n,p,q \in \mathbb{N}^*$, then $a_p \cdot a_q = a_m \cdot a_n$; if $2m=p+q$, then $a_m^2 = a_p \cdot a_q$;

② a_n, a_{n+m}, a_{n+2m}, $\cdots$ will form a geometric sequence with a common ratio of r^m;

③ If the common ratio of (a_n) is r, then the sequence $\left(\frac{1}{a_n}\right)$ is a geometric sequence with a common ratio of $\frac{1}{r}$;

④ If (a_n) and (b_n) are both geometric sequences, then $(a_n b_n)$ is also a geometric sequence;

⑤ If the common ratio is:

positive, then the terms will all be the same sign as the initial term;

negative, then the terms will alternate between positive and negative.

Let S_n denote the n^{th} partial sum of a geometric sequence whose first term is a_1, and the common ratio is r. Then $S_n = \begin{cases} na_1 (r=1) \\ \frac{a_1(1-r^n)}{1-r} (r \neq 1) \end{cases}$.

Geometric Sequence partial sums have the following properties:

① $(S_m, S_{2m}-S_m, S_{3m}-S_{2m}, \cdots)$ is a geometric sequence with a common ratio of r^m.

② Let $k \in \mathbb{N}^*$. Then, $\frac{\sum_{i=1}^{k} a_{2i}}{\sum_{i=1}^{k} a_{2i-1}} = r$.

Fixed Point Theorem

Consider a recurrence relation of the form $a_{n+2} = pa_{n+1} + qa_n$, where p, q are nonzero constant coefficients.

We say that $r^2 = pr + q$ is the characteristic equation for this recurrence relation. Solve for r to obtain the two roots α, β.

Different solutions are obtained depending on the nature of the roots:

① If these roots are distinct, we have the general solution $a_n = A\alpha^n + B\beta^n$;

② If they are identical ($p^2+4q=0$), we have $a_n = (An+B)\alpha^n$.

The roots of the function $f(x)=x$ are called the fixed points of $f(x)$.

① Let $f(u)=au+b$, where $a\neq 0,1$. If p is a fixed-point of f, and (u_n) satisfies that $u_n=f(u_{n-1})$ for all $n\geqslant 2$, then $u_n-p=a(u_{n-1}-p)$, i.e. (u_n-p) is a geometric sequence with common ratio a.

② Let $f(u)=\frac{au+b}{cu+d}$, where $c\neq 0, ad-bc\neq 0$. Suppose that (u_n) satisfies that $u_n=f(u_{n-1})$ for all $n\geqslant 2$, and $u_1\neq f(u_1)$.

If f has two distinct fixed-points p and q, then $\frac{u_n-p}{u_n-q}=k\cdot\frac{u_{n-1}-p}{u_{n-1}-q}$, where $k=\frac{a-pc}{a-qc}$.

If f has a unique fixed-point p, then $\frac{1}{u_n-p}=\frac{1}{u_{n-1}-p}+k$, where $k=\frac{2c}{a+d}$.

Iterated Functions

Let X be a set, and $f: X\to X$ be a function. Let $f^{(0)}(x)=x$, $f^{(1)}(x)=f(x)$, $f^{(2)}(x)=f(f(x))$, …, $f^{(n)}(x)=f(f^{(n-1)}(x))$.

The iterates of f are $f(x)$, $f(f(x))$, $f(f(f(x)))$, …, and we say that $f^{(n)}(x)$ is the n-th iterate of f, where n is a non-negative integer.

Methods of Finding the Iterative Formula:

1. Direct Computation

If $f(x)=x+a$, then $f^{(n)}(x)=x+na$.

If $f(x)=ax$, then $f^{(n)}(x)=a^n x$.

If $f(x)=ax^2$, then $f^{(n)}(x)=a^{2^n-1}\cdot x^{2n}$.

If $f(x)=\frac{x}{x+1}$, then $f^{(n)}(x)=\frac{x}{nx+1}$.

2. Mathematical Induction

Write out the first few iterations, and guess the iterative formula by observation. Then prove your answer correct by mathematical induction.

3. Using a Bridge Function

Given two functions $f(x)$ and $g(x)$, if there exists an invertible function $\varphi(x)$ such that $f(x)=\varphi^{-1}(g(\varphi(x)))$, then we say that $f(x)$ and $g(x)$ are similar with respect to $\varphi(x)$. We denote this similarity relation as $f\sim g$, and the function $\varphi(x)$ is called the bridge function.

If $f(x)$ and $g(x)$ are similar with respect to $\varphi(x)$, then $f(x) = \varphi^{-1}(g(\varphi(x)))$, and

$f^{(2)}(x) = f(f(x)) = \varphi^{-1}(g(\varphi(f(x)))) = \varphi^{-1}(g(\varphi(\varphi^{-1}(g(\varphi(x)))))) = \varphi^{-1}\left(g^{(2)}(\varphi(x))\right)$.

By induction, we can show that $f^{(n)}(x) = \varphi^{-1}\left(g^{(n)}(\varphi(x))\right)$.

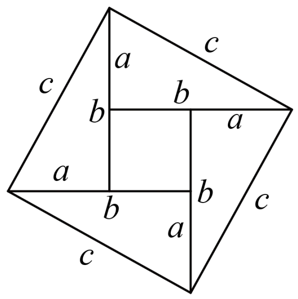
In this way, we can easily translate the n-th interate of f to the n-th iterate of g.

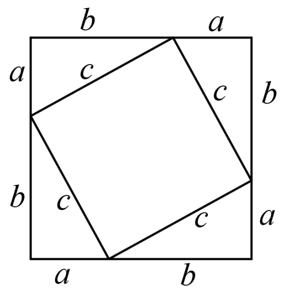
Geometry

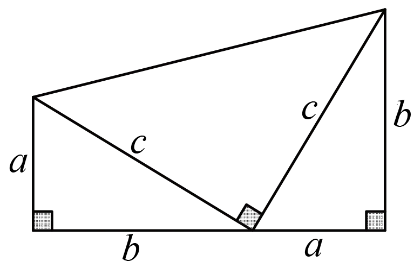
Pythagorean Theorem and its Converse

① Pythagorean Theorem: In a right triangle, the sum of the squares of the two legs is equal to the square of the hypotenuse.

② Converse of Pythagorean Theorem: In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.



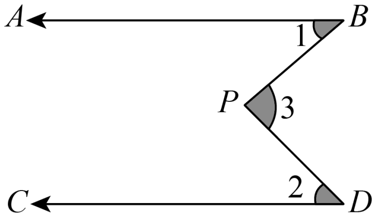




The figures above illustrate some classical proofs of the Pythagorean Theorem.

Trotter Model and Pencil Model

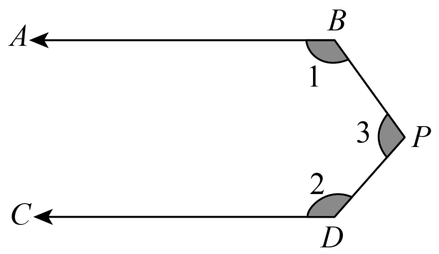
Given that two lines $AB//CD$, P is a point between AB and CD. Connecting BP and DP, we get the following figure containing three angles:



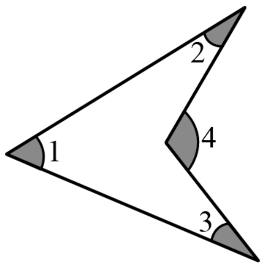
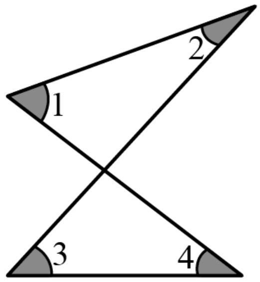
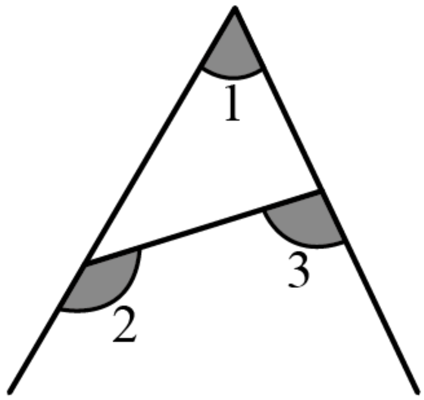
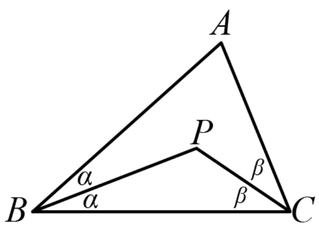
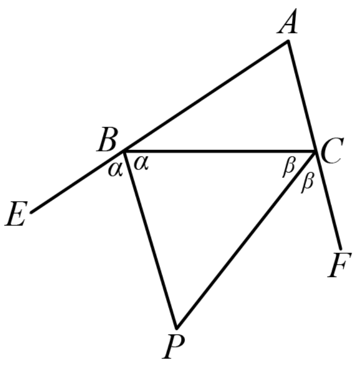
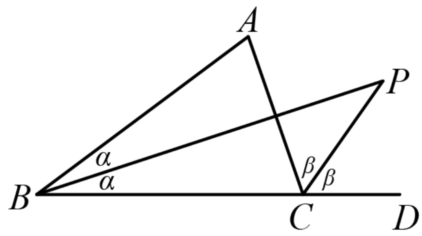
We can conclude that $\angle 3 = \angle 1 + \angle 2$, and we can apply this conclusion in many problems that require us to solve for angles.

To create a name for the model, notice that the model is very similar to the pig's feet, so we can call it a **Trotter Model**.

Alternatively, if point P is to the right of points B and D, we can conclude that $\angle 1+\angle 2+\angle 3=360^{\circ}$, and the model looks like a pencil, so we can call it a **Pencil Model**.



Useful Models for Angle Measurement

Dart Model	"8" Model	"A" Model
		
$\angle 4=\angle 1+\angle 2+\angle 3$	$\angle 1+\angle 2=\angle 3+\angle 4$; if $\angle 1=\angle 3$, then $\angle 2=\angle 4$	$\angle 2+\angle 3=\angle 1+180^{\circ}$
Interior Angle Bisector Model	Exterior Angle Bisector Model	Interior & Exterior Angle Bisector Model
		
$\angle P=90^{\circ}+\frac{1}{2}\angle A$	$\angle P=90^{\circ}-\frac{1}{2}\angle A$	$\angle P=\frac{1}{2}\angle A$

Common Ratios and Formulae

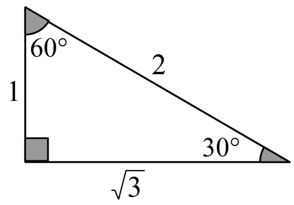
- Isosceles right triangles: the side lengths have the ratio of $1:1:\sqrt{2}$;
- Equilateral triangles: the altitude and side length have the ratio $\sqrt{3}:2$. If the side length is a, then the area of the equilateral triangle is $\frac{\sqrt{3}a^2}{4}$;
- Isosceles triangles with an apex angle of 120°: the side lengths satisfy the ratio of $1:1:\sqrt{3}$. With leg length a, then the area of such a triangle is $\frac{\sqrt{3}a^2}{4}$;
- Golden triangles: triangles with three interior angles equal to 36°, 72°, and 72°, respectively. The side lengths satisfy the ratio of $1:1:\frac{\sqrt{5}-1}{2}$.

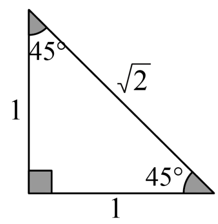
Special Right Triangles

Two types of special right triangles:

① In a right triangle, if the interior angles are 30°, 60°, 90°, respectively, then the corresponding side ratio is $1:\sqrt{3}:2$.

② In a right triangle, if the interior angles are 45°, 45°, 90°, respectively, then the corresponding side ratio is $1:1:\sqrt{2}$.



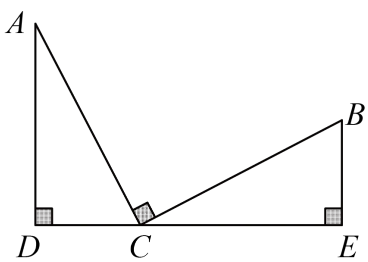


Triple Perpendicular Congruence Model

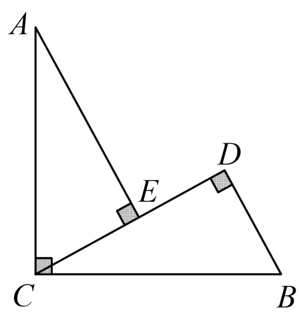
(1) Refer to Figure ①, $AC=BC$, $AC\perp BC$, DE passes through point C. $AD\perp DE$ at D, $BE\perp DE$ at E. Then we can prove $\triangle ACD\cong\triangle CBE$.

(2) Refer to Figure ②, $AC=BC$, $AC\perp BC$, CD passes through C. $AE\perp CD$ at E, $BD\perp CD$ at D. Then we can prove $\triangle ACE\cong\triangle CBD$.

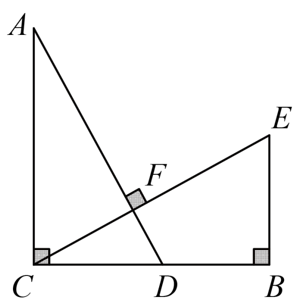
(3) Refer to Figure ③, $AC=BC$, $AC\perp BC$. $EB\perp BC$ at B, $AD\perp CE$ and intersect with CE at F. Then we can prove $\triangle ACD\cong\triangle CBE$.



①

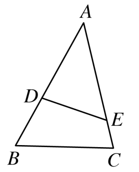
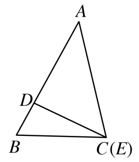
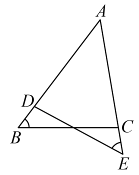
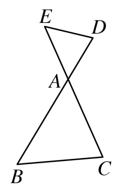


②



③

Common Models in Similar Triangles

<table>
<tr>
<td>Reversed "A" Model
If E is a point on side $\overline{AC}$ and $\angle AED=\angle ABC$, then we have
$\triangle AED\sim\triangle ABC$, thus $\frac{AE}{AB}=\frac{AD}{AC}=\frac{DE}{BC}$, $AE\cdot AC=AD\cdot AB$.</td>
<td>
</td>
</tr>
<tr>
<td>Reversed "A" Model with Common Side
When segment CD is moved down so that E is coincides with C, if
$\angle ACD=\angle ABC$, we have $\triangle AC(E)D\sim\triangle ABC$, thus $\frac{AC}{AB}=\frac{AD}{AC}=\frac{DC}{BC}$,
$AC^2=AD\cdot AB$.
Especially, when $\angle ADC=\angle ACB=90^\circ$, we have $\triangle DCB\sim\triangle CAB\sim\triangle DAC$
and $CD^2=AD\cdot BD$, $AC^2=AD\cdot AB$, $BC^2=BD\cdot AB$.</td>
<td>
</td>
</tr>
<tr>
<td>Dart Model
When $\angle E=\angle B$,
$\triangle ADE\backsim\triangle ACB$.
We have $\frac{AE}{AB}=\frac{AD}{AC}=\frac{DE}{CB}$,
then we can conclude that $AD\cdot AB=AE\cdot AC$.</td>
<td>
</td>
</tr>
<tr>
<td>Reversed "8" Model
When $\angle AED=\angle ABC$, we have $\frac{AE}{AB}=\frac{AD}{AC}=\frac{DE}{BC}$, $AE\cdot AC=AD\cdot AB$.</td>
<td>
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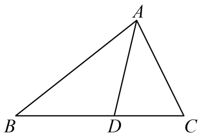
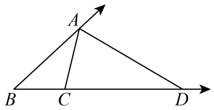
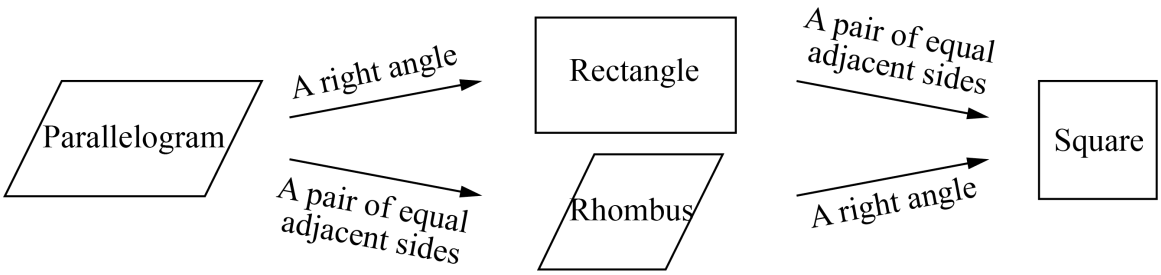
Angle Bisector Theorem

Interior Angle Bisector Theorem 	In $\triangle ABC$, AD is the bisector of $\angle BAC$. Then we have $\frac{AB}{AC}=\frac{BD}{CD}$.
Exterior Angle Bisector Theorem 	In $\triangle ABC$, $AB \neq AC$. The bisector of the exterior angle of $\angle BAC$ intersects with the extension of BC at D. Then we have $\frac{AB}{AC}=\frac{BD}{CD}$.

Differences and Connections between Parallelograms, Rectangles, Rhombi and Squares

Elements	Characters	Parallelogram	Rectangle	Rhombus	Square
Opposite Side	Positional	Parallel	Parallel	Parallel	Parallel
	Length	Equal	Equal	Equal	Equal
Adjacent Side	Positional	Intersect	Perpendicular	Intersect	Perpendicular
	Length	Not necessarily equal	Not necessarily equal	Equal	Equal
Angle	Opposite	Equal	Equal	Equal	Equal
	Adjacent	Supplementary	Supplementary and equal	Supplementary	Supplementary and equal
Diagonal	Positional	Intersect	Intersect	Perpendicular	Perpendicular
	Length	Not necessarily equal	Equal	Not necessarily equal	Equal
Symmetry		Centrosymmetric	Centrosymmetric and axisymmetric	Centrosymmetric and axisymmetric	Centrosymmetric and axisymmetric



Circles and Regular Polygons

Defition of regular polygons: Polygons with all equal sides and angles are called regular polygons. If a regular polygon has n sides, then it is called a regular n-side polygon.

Concepts of regular n-side polygons

Center of a regular n-side polygon: The center of the circumscribed (or inscribed) circle of the polygon.

Radius of a regular n-side polygon: The radius of its circumscribed circle (R).

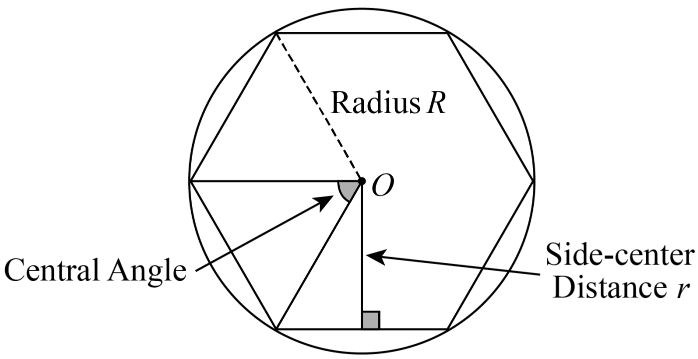
Side-center distance of regular n-side polygon: The radius of its inscribed circle (r).

Central angle of a regular n-side polygon: The corresponding central angle between the circumscribed circle and a side of polygon. The measure is $\frac{360^\circ}{n}$.

The measure of interior angle of a regular n-side polygon is $\frac{(n-2)\cdot 180^\circ}{n}$.

The measure of exterior angle of a regular n-side polygon is $\frac{360^\circ}{n}$.

The measures of a central angle and an exterior angle of a regular n-side polygon are equal to each other.



Concepts and Symmetricity of regular n-side polygons

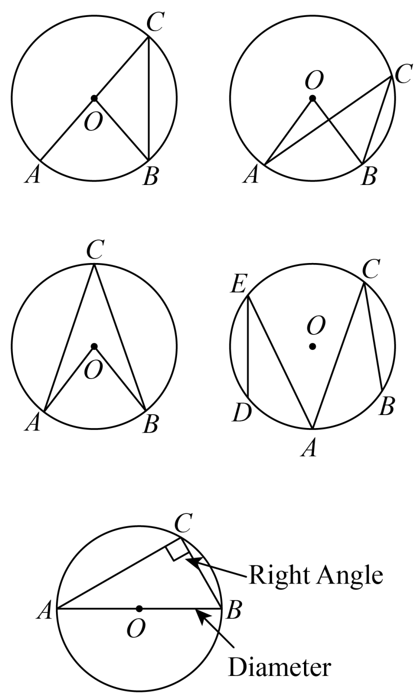
Regular n-side polygons are all axisymmetric figures. A regular n-side polygon has n axes of symmetry, and all of them intersect at the center.

Regular n-side polygons are all rotational symmetric figures. When n is odd, the regular polygon is not centrosymmetric; when n is even, the regular polygon is centrosymmetric, with the the center of the polygon as the symmetric center.

Basics of Circles

<table>
<tr><th>Knowledge Points</th><th>Examples</th></tr>
<tr><td>Inscribed Angle Theorem: an inscribed angle is half of a central angle subtended by the same arc.

Corollary 1: Inscribed angle subtended by the same arc (or congruent arc) are equal. Inscribed angle subtended by the equal chord in the same circle (or congruent circles) are equal or supplementary.

Corollary 2: Inscribed angle subtended by semicircle (or diameter) are right angle. If an inscribed angle is a right angle, then the chord subtends it is a diameter.</td><td>
</td></tr>
</table>

Perpendicular Diameter Theorem

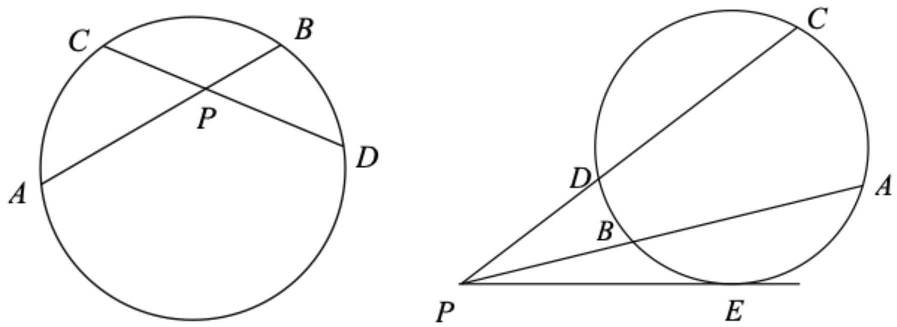
Theorem: If two chords in a circle are congruent then their intercepted arcs are congruent.

Converse: If two arcs are congruent then their corresponding chords are congruent.

Power of a Point Theorem

The **Power of a Point Theorem** is a relationship that holds between the lengths of the line segments formed when two lines intersect a circle and each other.

Given a point P and a circle, pass two lines through P that intersect the circle in points A and B and, respectively, C and D. Then $PA \cdot PB = PC \cdot PD$.



As shown in the figures, the point P may lie either inside or outside the circle, and the line through A and B (or that through C and D, or both) may be tangent to the circle, in which case A and B coalesce into a single point E. In all cases, the above statement holds, and it is known as the power of a point theorem.

When the point P is inside the circle, as shown on the left, the theorem is also known as the **Theorem of Intersecting Chords** (or the **Intersecting Chords Theorem**).

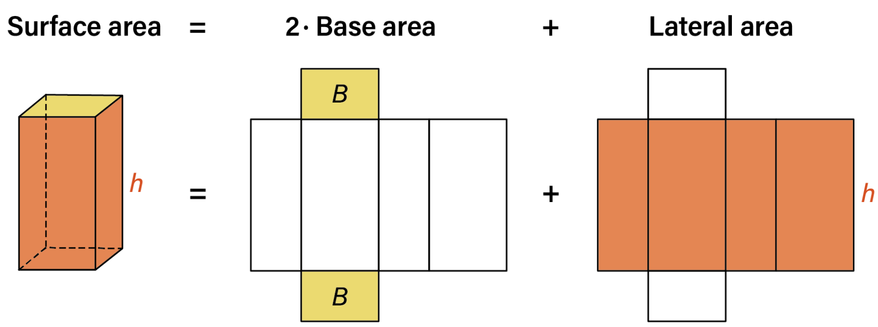
When the point P is outside the circle, as shown on the right, the theorem becomes the **Theorem of Intersecting Secants** (or the **Intersecting Secants Theorem**).

Prisms, Pyramids, and Regular Polyhedra

1. A **prism** is a three-dimensional figure with two identical and parallel bases that are polygons and the other faces are rectangles. A prism is identified by the shape of its base.

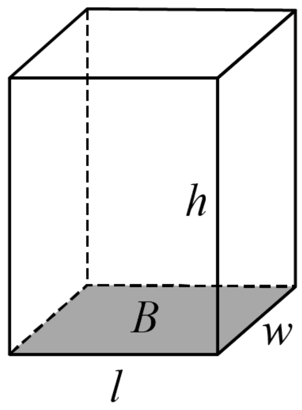
Surface Area $= 2\times$ Base Area$+$ Lateral Area.

The surface area of a prism is the sum of the areas of the lateral faces (faces that are not bases) and bases.

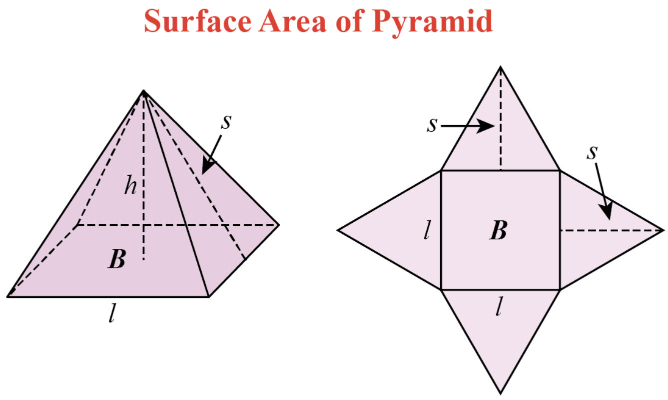


Volumes of prisms:

$V = Bh$



2. A **pyramid** is a three-dimensional figure whose base is a polygon and whose other faces are triangles that meet at a point called the **apex**. A pyramid is identified by the shape of its base.



l: the side length of the base;

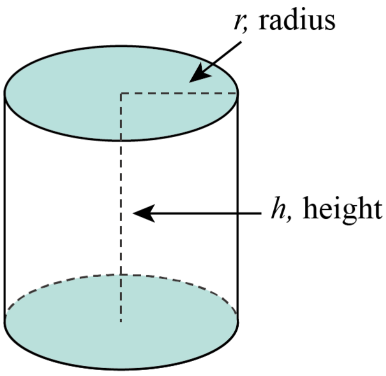
s: the slant height (if all of the slant heights are the same);

The surface area is calculated by $l^2 + 2ls$;

The volume is calculated by $\frac{1}{3} \times B \times h$.

Cylinders, Cones, and Spheres

1. A cylinder is a solid figure with two congruent circular bases that lie in parallel lines.

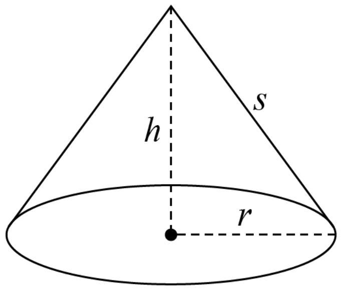


Base area $B=\pi r^2$

Surface area $S=2B+2\pi rh=2\pi r^2+2\pi rh$

Volume $V=Bh=\pi r^2h$

2. A **cone** is a solid figure with a circular base. As we can see, the lateral area of a cone is a sector.



The **slant height** s of a cone is the distance between any point on the edge of the base and the vertex.

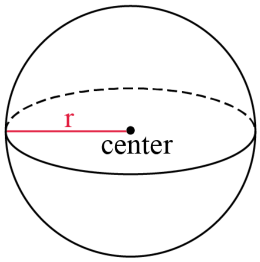
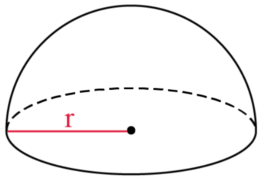
Slant height $s=\sqrt{r^2+h^2}$

Base Area $B=\pi r^2$

Surface area $S=B+\pi rs=\pi r^2+\pi rs$, where slant height $s=\sqrt{r^2+h^2}$

Volume $V=\frac{1}{3}\pi r^2h$

3. Just as a circle is the set of all points in a plane that are the same distance from a given point, a **sphere** is the set of all points in space that are equidistant from a given point. Most balls and globes are examples of spheres.

Sphere		Hemisphere	
	$V=\frac{4}{3}\pi r^3$ $SA=4\pi r^2$		$V=\frac{1}{2}(\frac{4}{3}\pi r^3)=\frac{2}{3}\pi r^3$ $SA=\frac{1}{2}(4\pi r^2)+\pi r^2=3\pi r^2$

Number Theory

Divisibility Rules

1. Numbers divisible by 2 have their unit digit as a multiple of 2, which can be $0, 2, 4, 6, 8$.
2. Numbers divisible by 3 have the sum of their digits as a multiple of 3.
3. Numbers divisible by 4 have their last two-digit number as a multiple of 4.
4. Numbers divisible by 5 have a unit digit of either 0 or 5.
5. (Infrequently used) Numbers divisible by 7 can be identified by taking away the last three digits and then subtracting the three-digit number to the remaining number. If the difference is a multiple of 7, the original number is divisible by 7.
6. Numbers divisible by 8 have their last three-digit number as a multiple of 8.
7. Numbers divisible by 9 have the sum of their digits as a multiple of 9.
8. Numbers divisible by 11 have their difference between the sum of odd-positioned digits and even-positioned digits being divisible by 11.
9. (Infrequently used) Numbers divisible by 13 can be identified by taking away the unit digit and then adding four times the unit digit to the remaining number. If the sum is a multiple of 13, the original number is divisible by 13.

GCD and LCM

Least Common Multiple (LCM):

A multiple that is shared by two or more integers is called their **common multiple**. Among these, the smallest common multiple, excluding 0, is referred to as the **least**

common multiple of those integers.

Greatest Common Divisor (GCD):

A divisor that is shared by two or more integers is called their **common divisor**. Among these, the largest divisor, excluding 0, is known as the **greatest common divisor** of those integers.

Least Common Multiple $\times$ Greatest Common Divisor $=$ Product of the two integers

Euclid's Lemma

If a prime number p divides the product ab of two integers a and b, then p must divide at least one of those integers a and b.

In general, if a prime integer p divides the product $a_1 \ldots a_k$, then p must divide at least one of the integers $a_1, a_2, \ldots, a_k$.

Number of Factors

Consider positive integer $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. We use $\tau(n)$ to represent the number of positive factors of n ($d(n)$ in some texts), then

$\tau(n) = \tau\left(p_1^{\alpha_1}\right) \tau\left(p_2^{\alpha_2}\right) \cdots \tau\left(p_k^{\alpha_k}\right) = (\alpha_1 + 1) \cdots (\alpha_k + 1) = \prod_{i=1}^{k} (\alpha_i + 1)$.

Properties of Congruence Modulo

Given a positive integer m, if two integers a and b satisfy the condition that $a - b$ is divisible by m, meaning that $\frac{a-b}{m}$ results in an integer, then the integers a and b are said to be **congruent modulo** m, denoted as $a \equiv b \pmod{m}$.

Reflexivity: $a \equiv a \pmod{m}$.

Symmetry: If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$.

Transitivity 1: If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

Transitivity 2: If $a \equiv b \pmod{m}$, then $a \equiv b \pmod{i \cdot m}$ for $i \in \mathbb{Z}^+$.

Transitivity 3: If $a \equiv b \pmod{m}$ and d divides m, then $a \equiv b \pmod{d}$.

Addition of Congruences: If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$.

Multiplication of Congruences: If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a \cdot c \equiv b \cdot d \pmod{m}$.

Corollary: If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$ for all $n \in \mathbb{Z}^+$.

Linear Operations: If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then:

(1): $a \pm c \equiv b \pm d \pmod{m}$.

(2): $a \cdot c \equiv b \cdot d \pmod{m}$.

Modular Division: If $ac \equiv bc \pmod{m}, (m, c) = d$, then $a \equiv b \pmod{m/d}$; particularly, when $(m, c) = 1, a = b \pmod{m}$.

Modular Multiplication: If $a = b \pmod{m}$, then $a^k = b^k \pmod{m^k}$ for any positive integer k.

Corollary: If $a \equiv b \pmod{m}$ and d is a positive divisor of both a and b as well as m, then $\frac{a}{d} \equiv \frac{b}{d} \pmod{\frac{m}{d}}$.

Fermat's little theorem: if p is a prime number, then for any integer a, the number $a^p - a$ is an integer multiple of p.

$a^p \equiv a \quad (\bmod p)$

Euler's theorem: if n and a are coprime positive integers, and $\varphi(n)$ is Euler's totient function, then a raised to the power $\varphi(n)$ is congruent to 1 modulo n.

$a^{\varphi(n)} \equiv 1 \quad (\bmod n)$.

Place Value and Number Base

Every digit in a number has a **place value**. Place value can be defined as the value represented by a digit in a number on the basis of its position in the number.

For example, the place value of 3 in 2315 is 3 hundreds or 300. The place value of 5 in 5432 is 5 thousands or 5000.

Given an m-digit positive integer A with $a_{m-1}, a_{m-2}, \ldots, a_0$ being its digits from left to right, we denote this number by $A = \overline{a_{m-1}a_{m-2}\cdots a_0}$, where $a_{m-1} \neq 0$.

The decimal number system we use has 10 digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, so it is called **base-10 number** system.

We can write a number in decimal number system as

$A=a_{m-1}\cdot 10^{m-1}+a_{m-2}\cdot 10^{m-2}+\cdots+a_1\cdot 10+a_0$, where $a_i\in\{0,1,2,\cdots,9\}$ for $i=0,1,2,\cdots,m-1$ and $a_{m-1}\neq 0$.

To convert a number from decimal to base-p number system, we perform the following steps.

step 1: Divide the decimal number to be converted by p.

step 2: Use the remainder from step 1 as the rightmost digit of the new base number.

step 3: Divide the quotient from step 1 by the new base.

step 4: Record the remainder from step 3 as the next digit to the left of the new base number.

To convert a number in base-p to a decimal number, starting from the rightmost digit of the number, we multiply the i-th digit by p^{i-1} and add them up.

For instance, $11101_2=\left(\left(1\times 2^4\right)+\left(1\times 2^3\right)+\left(1\times 2^2\right)+\left(0\times 2^1\right)+\left(1\times 2^0\right)\right)_{10}$

Linear Diophantine Equations (LDE)

A Linear Diophantine equation (LDE) is an equation with 2 or more integer unknowns and the integer unknowns are at most degree of 1. Linear Diophantine equation in two variables takes the form of $ax+by=c$, where $x,y\in\mathbb{Z}$ and a,b,c are integer constants.

1. If $\gcd(a,b)\nmid c$, then there is no integer solutions to $ax+by=c$. For instance, the LDE $2x+4y=5$ has no integer solutions since $\gcd(2,4)=2\nmid 5$.

2. If $\gcd(a,b)=1$, then $ax+by=1$ and $ax+by=c$ both have integer solutions. If (x_0,y_0) is a solution to $ax+by=1$, then (cx_0,cy_0) is a solution to $ax+by=c$.

3. $ax+by=\gcd(a,b)$ has integer solutions.

4. If $\begin{cases}x=x_0\\ y=y_0\end{cases}$ is an integer solution to $ax+by=c$, then $\begin{cases}x=x_0+bu\\ y=y_0-au\end{cases}$ (where u is an arbitrary integer) is also an integer solution to $ax+by=c$.

Combinatorics and Counting

Addition and Multiplication Rules

Addition Rule: There are n ways to complete a task. For the first way, there are m_1 different methods; for the second way, there are m_2 different methods; $\cdots$; for the n^{th} way, there are m_n different methods. Therefore, there are $N = m_1 + m_2 + \cdots + m_n$ different methods to complete the task.

Multiplication Rule: We need n steps to complete a task. There are m_1 different methods to complete the first step; there are m_2 different methods to complete the second step; $\cdots$; there are m_n different methods to complete the n^{th} step. Therefore, there are $N = m_1 \times m_2 \times \cdots \times m_n$ different methods to complete the task.

Permutation

A permutation is an arrangement of all or part of a given set of objects, with regard to the order of the arrangement.

Given a set of n distinguishable objects, the number of ways to select m $(m \leqslant n)$ objects from the set in order is denoted by ${}_nP_m = n(n-1)(n-2)\cdots(n-m+1) = \dfrac{n!}{(n-m)!}$, which is read as "$n$ permute m".

In particular, if we select all n objects for a permutation, then the action is called a **full permutation** of n objects, which is just the factorial of n (define $0! = 1$):

${}_nP_n = n \times (n-1) \times \cdots \times 2 \times 1 = n!$

Combination

A combination is an arrangement of all or part of a given set of objects where the order of the selection does not matter.

Given a set of n distinguishable objects, the number of ways to select m $(m \leqslant n)$ objects from the set without regard to order is

${}_nC_m = \dfrac{n(n-1)(n-2)\cdots(n-m+1)}{m!} = \dfrac{{}_nP_m}{{}_mP_m} = \dfrac{n!}{m!(n-m)!}$, which is read as "$n$ choose m ".

Basic properties of combination (define ${}_nC_0 = 1$):

① ${}_{n}C_{m} = {}_{n}C_{n-m}$;

② ${}_{n+1}C_{m} = {}_{n}C_{m} + {}_{n}C_{m-1}$.

Some useful conclusions:

- The number of distinct ways to arrange m objects selected from a set n distinguishable objects along a line is ${}_{n}P_{m}$.
- The number of distinct ways to choose m objects from a set of n distinguishable objects is ${}_{n}C_{m}$.
- The number of distinct ways to arrange m objects selected with replacement from a set of n distinguishable objects is n^{m}.
- The number of distinct ways to choose m objects with replacement from a set of n distinguishable objects is ${}_{n+m-1}C_{m}$.
- Given a set of n objects, we divide the set into k groups and assume that objects in the same group are indistinguishable and objects in different groups are distinguishable. Suppose that there are n_i objects in i-th group where $i = 1, 2, \ldots, k$ and $n_1 + n_2 + \cdots + n_k = n$. Then, the number of distinct ways to permute such a set of n objects is $\frac{n!}{n_1!n_2!\ldots n_k!}$.
- The number of distinct ways to arrange m objects from n distinguishable objects and form a circular arrangement is $\frac{{}_{n}P_{m}}{m} = \frac{n!}{(n-m)! \cdot m}$.

Stars and Bars Method

Stars and bars method is commonly used in combinatorics to solve counting problems, such as to count the number of ways to put n indistinguishable balls into k distinguishable bins with no bin being empty.

To solve this problem, we first draw n stars in a line. We then indicate the placement of the stars into different bins by placing $k-1$ bars between the stars. Since no bin is allowed to be empty, we can place at most one bar in each gap. The problem is reduced to selecting $k-1$ gaps from the $n-1$ gaps to place a bar. Thus, the number of ways to put n indistinguishable balls into k distinguishable bins with no bin being empty is ${}_{n-1}C_{k-1}$.

Below is an example of putting 6 indistinguishable balls into 3 distinguishable bins.

$***|**|*$ means to place 3 balls in the first bin, 2 balls in the second bin and 1 ball in the last bin.

$**|**|**$ means to place 2 balls in each of the three bins.

Probability

In probability, an **elementary event** (also known as a sample point) represents a single outcome in the sample space.

Elementary events have the following properties:

(1) Any two elementary events are disjoint, i.e. they have no outcomes in common.

(2) Any event (except for impossible events) can be written as a sum of some elementary events.

Classical probability is an approach to understanding the probability based on the assumptions that any experiment has only finitely many possible outcomes and that each possible outcome is equally likely to occur.

The **probability** of a simple event happening is the number of times the event can happen, divided by the number of possible outcomes.

The **probability** of event A is $P(A)=\frac{m}{n}$, where m is the number of elementary events that A contains and n is the number of possible outcomes in the experiment.

Expected Value

The **expected value** or **expectation** (also called the **mean**) of a random variable X is the weighted average of the possible values of X, weighted by their corresponding probabilities.

Suppose that the possible values of a discrete random variable X are $x_1, x_2, \ldots, x_n$, whose corresponding probabilities are $p_1, p_2, \ldots, p_n$, respectively. The expectation of X is $E(X)=\sum_{i=1}^{n} x_i p_i$.

The expectation of a random variable is the value that we would expect to see on average after repeated observation of the random process.

The Pigeonhole Principle

The **pigeonhole principle**, also known as **Dirichlet's box principle** or **Dirichlet's drawer principle**, states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

- For natural numbers n and m, if n objects are distributed among m sets, then the pigeonhole principle asserts that at least one of the sets will contain at least $\left\lceil \frac{n}{m} \right\rceil$ objects. Similarly, at least one of the sets will contain no more than $\left\lfloor \frac{n}{m} \right\rfloor$ objects.
- If $m_1 + m_2 + \cdots + m_n + 1$ items are put into n containers, where $m_1, m_2, \ldots, m_n$ are positive integers, then either the first box contains at least $m_1 + 1$ objects, or the second box contains at least $m_2 + 1$ objects, $\ldots$, or the n-th box contains at least $m_n + 1$ objects.
- If $m_1 + m_2 + \cdots + m_n - 1$ items are put into n containers, where $m_1, m_2, \ldots, m_n$ are positive integers, then either the first box contains at most $m_1 - 1$ objects, or the second box contains at most $m_2 - 1$ objects, $\ldots$, or the n-th box contains at most $m_n - 1$ objects.

Principle of inclusion and exclusion

1. Principle of inclusion and exclusion (PIE):

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

2. Complementary form:

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right| = |I| - \sum_{i=1}^{n} |A_i| + \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$$

in which I is the universal set, $A_1 \cup A_2 \cup \cdots \cup A_n \subseteq I$.

3. Common ternary set exclusion principles:

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap C| + |A \cap B| + |B \cap C|) + |A \cap B \cap C|$$

$$\left|\overline{A} \cap \overline{B} \cap \overline{C}\right| = |S| - (|A| + |B| + |C|) + (|A \cap C| + |A \cap B| + |B \cap C|) - |A \cap B \cap C|$$

Graph Theory Basics

A diagram, made up from points and lines that connect some pairs of those points, is called a **graph**. The dots are called the **vertices** of the graph, and the lines are called the **edges** of the graph.

A vertex is a dot in the graph that could represent a person, a location, etc.

Edges connect pairs of vertices. An edge can represent a connection between two locations, e.g. a bridge connecting two islands.

A graph in which each edge is assigned a direction is called a **directed graph** or **digraph**.

The **complete graph** on n vertices, denoted by K_n, is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge.

A **k-partite graph** is a graph whose vertices are (or can be) partitioned into k different independent sets. When $k = 2$, these are the bipartite graphs.

A **complete k-partite graph** is a k-partite graph in which there is an edge between every pair of vertices from different independent sets.

For instance, a graph on six vertices divided into two subsets of size three each, with edges joining every vertex in one subset to every vertex in the other subset, is called a complete bipartite graph. We denote this graph by $K_{3,3}$ and it is also known as the utility graph.