MAA AMC

American Mathematics Competitions

MAA American Mathematics Competitions

25th Annual

AMC 10 B

Tuesday, November 14, 2023

INSTRUCTIONS

1. DO NOT TURN TO THE NEXT PAGE UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
2. This is a 25-question multiple-choice competition. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, compasses, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your competition manager will ask you to record your name on the answer sheet.
8. You will have 75 minutes to complete the competition once your competition manager tells you to begin.
9. When you finish the competition, sign your name in the space provided on the answer sheet and complete the demographic information questions on the back of the answer sheet.

The problems and solutions for this AMC 10 B were prepared
by the MAA AMC 10/12 Editorial Board under the direction of
Gary Gordon and Carl Yerger, co-Editors-in-Chief.

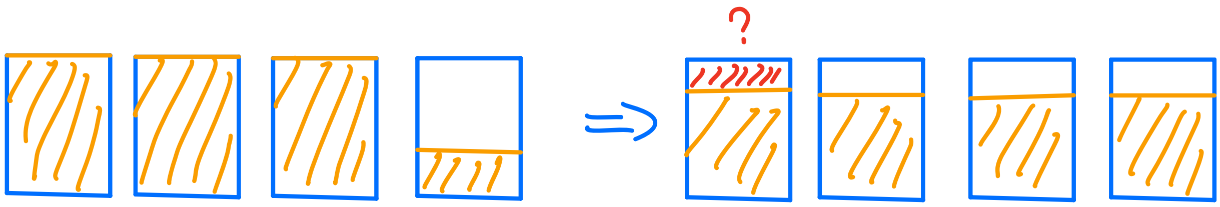
The MAA AMC office reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction, or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

Students who score well on this AMC 10 will be invited to take the 42nd annual American Invitational Mathematics Examination (AIME) on Thursday, February 1, 2024, or Wednesday, February 7, 2024. More details about the AIME can be found at maa.org/AIME.

1. Mrs. Jones is pouring orange juice into four identical glasses for her four sons. She fills the first three glasses completely but runs out of juice when the fourth glass is only $\frac{1}{3}$ full. What fraction of a glass must Mrs. Jones pour from each of the first three glasses into the fourth glass so that all four glasses will have the same amount of juice?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{2}{9}$



$$1-\left(3\tfrac{1}{3}\div 4\right)=1-\frac{10}{3}\div 4=1-\frac{5}{6}=\frac{1}{6}$$

2. Carlos went to a sports store to buy running shoes. Running shoes were on sale, with prices reduced by 20% on every pair of shoes. Carlos also knew that he had to pay a 7.5% sales tax on the discounted price. He had \$43. What is the original (before discount) price of the most expensive shoes he could afford to buy?

(A) \$46 **(B)** \$50 **(C)** \$48 **(D)** \$47 **(E)** \$49

Suppose the original price is x dollars per pair of shoes, so after discount, the price is $(1-20\%)x = 0.8x$ dollars, after tax, the price is $(1+7.5\%)\cdot 0.8x$ dollars.

so: $1.075\cdot 0.8x \le 43$

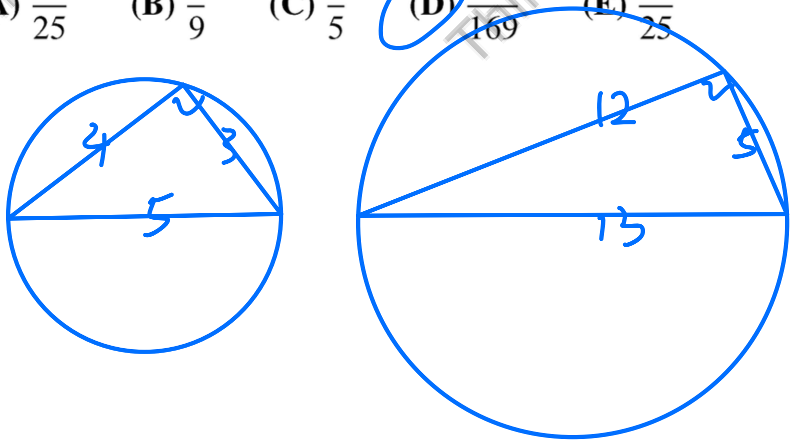
$0.86x \le 43$

$x \le 50$

$\therefore$ the maximum value of x is 50.

3. A $3-4-5$ right triangle is inscribed in circle A, and a $5-12-13$ right triangle is inscribed in circle B. What is the ratio of the area of circle A to the area of circle B ?

(A) $\frac{9}{25}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{25}{169}$ **(E)** $\frac{4}{25}$



by the converse of Pythagorean theorem, these two triangles are right triangles.

So the circumcircle's diameter has the same length with the hypotenuse.

ratio of the area = (ratio of the radius)^2

$= \left(\frac{5}{13}\right)^2 = \frac{25}{169}$

4. Jackson's paintbrush makes a narrow strip with a width of 6.5 millimeters. Jackson has enough paint to make a strip 25 meters long. How many square centimeters of paper could Jackson cover with paint?

(A) 162,500 **(B)** 162.5 **(C)** 1,625 **(D)** 1,625,000 **(E)** 16,250

25 meters = 2500 centimeters

6.5 millimeters = 0.65 centimeters

$2500 \times 0.65 = 1625$

5. Maddy and Lara see a list of numbers written on a blackboard. Maddy adds 3 to each number in the list and finds that the sum of her new numbers is 45. Lara multiplies each number in the list by 3 and finds that the sum of her new numbers is also 45. How many numbers are written on the blackboard?

(A) 10 **(B)** 5 **(C)** 8 **(D)** 6 **(E)** 9

suppose there are x numbers on the blackboard and their sum are y.

so: $\begin{cases} y+3x=45 \\ 3y=45 \end{cases} \Rightarrow \begin{cases} x=10 \\ y=15 \end{cases}$

6. Let $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$. How many terms in the sequence $L_1, L_2, L_3, \ldots, L_{2023}$ are even?

(A) 673 **(B)** 1011 **(C)** 675 **(D)** 1010 **(E)** 674

Think Academy Summer Algebra 2 Honors Lesson 1 Assignment Q10.

Odd + Odd = Even

Even + Odd = Odd.

so we have the pattern:

L_1 Odd

L_2 Odd

L_3 Even

L_4 Odd

L_5 Odd

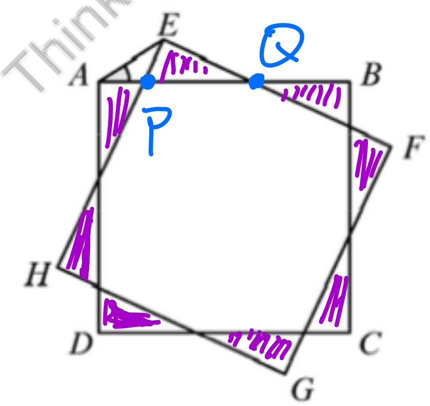
L_6 Even

...

for L_n. if and only if $3|n$, $2|L_n$.

$\left\lfloor \frac{2023}{3} \right\rfloor = 674$

7. Square $ABCD$ is rotated 20° clockwise about its center to obtain square $EFGH$, as shown below. What is the degree measure of $\angle EAB$?



(A) 24° **(B)** 35° **(C)** 30° **(D)** 32° **(E)** 20°

$\angle EQP = 20^\circ$, $\angle PEQ = 90^\circ$, $\therefore \angle EPQ = 70^\circ$

$\because AP = EP$ (easy to see all the eight purple triangles are congruent)

$\therefore \angle EAP = \angle AEP = \dfrac{\angle EPQ}{2} = 35^\circ$

8. What is the units digit of $2022^{2023} + 2023^{2022}$?

(A) 7 **(B)** 1 **(C)** 9 **(D)** 5 **(E)** 3

n	1	2	3	4	5	6	7	...
Last digit of 2^n	2	4	8	6	2	4	8	...
Last digit of 3^n	3	9	7	1	3	9	7	...

$2022^{2023} \equiv 2^{2023} \equiv 2^3 \equiv 8 \pmod{10}$

$2023^{2022} \equiv 3^{2022} \equiv 3^2 \equiv 9 \pmod{10}$

$8+9=17$. $\therefore$ the last digit is 7

9. The numbers 16 and 25 are a pair of consecutive positive perfect squares whose difference is 9. How many pairs of consecutive positive perfect squares have a difference of less than or equal to 2023 ?

(A) 674 **(B)** 1011 **(C)** 1010 **(D)** 2019 **(E)** 2017

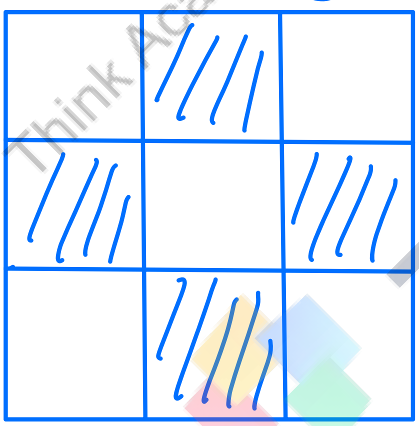
$n^2-(n-1)^2 \le 2023 \Rightarrow 2n-1 \le 2023$

$\therefore n \le 1012$. notice that $n-1>0$

$\therefore n \ge 2$. in total there are $1012-1=1011$

10. You are playing a game. A 2×1 rectangle covers two adjacent squares (oriented either horizontally or vertically) of a 3×3 grid of squares, but you are not told which two squares are covered. Your goal is to find at least one square that is covered by the rectangle. A "turn" consists of you guessing a square, after which you are told whether that square is covered by the hidden rectangle. What is the minimum number of turns you need to ensure that at least one of your guessed squares is covered by the rectangle?

(A) 3 **(B)** 5 **(C)** 4 **(D)** 8 **(E)** 6



If you are a player of final fantasy XIV who unlock wondrous tails, that should be a quite easy problem to you.

In fact, the problem is asking: How can you color a 3x3 white grid with the fewest black squares such that any two adjacent squares have at least one black square between them?

As the figure above shown, at least 4.

11. Suzanne went to the bank and withdrew \$800. The teller gave her this amount using \$20 bills, \$50 bills, and \$100 bills, with at least one of each denomination. How many different collections of bills could Suzanne have received?

(A) 45 **(B)** 21 **(C)** 36 **(D)** 28 **(E)** 32

we are soling Diophantine Equation:

$20x + 50y + 100z = 800 \Rightarrow 2x + 5y = 80 - 10z$

one special solution is $\begin{cases} x = 5 - 5z \\ y = 14 \end{cases}$

so general solution is $\begin{cases} x = 5 - 5z + 5k \\ y = 14 - 2k \end{cases}$ (k is integer)

$\because x, y, z \geq 1, \quad \therefore k \leq 6$ and $k \geq z$

$\therefore$ when $z = 1, k = 1 \sim 6$; $z = 2, k = 2 \sim 6$;

$\cdots\cdots$; $z = 6, k = 6$.

in total, $6 + 5 + 4 + 3 + 2 + 1 = \frac{6 \times 7}{2} = 21$.

12. When the roots of the polynomial

$$P(x) = (x-1)^1(x-2)^2(x-3)^3 \cdots (x-10)^{10}$$

are removed from the real number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is $P(x)$ positive?

(A) 3 **(B)** 7 **(C)** 6 **(D)** 4 **(E)** 5

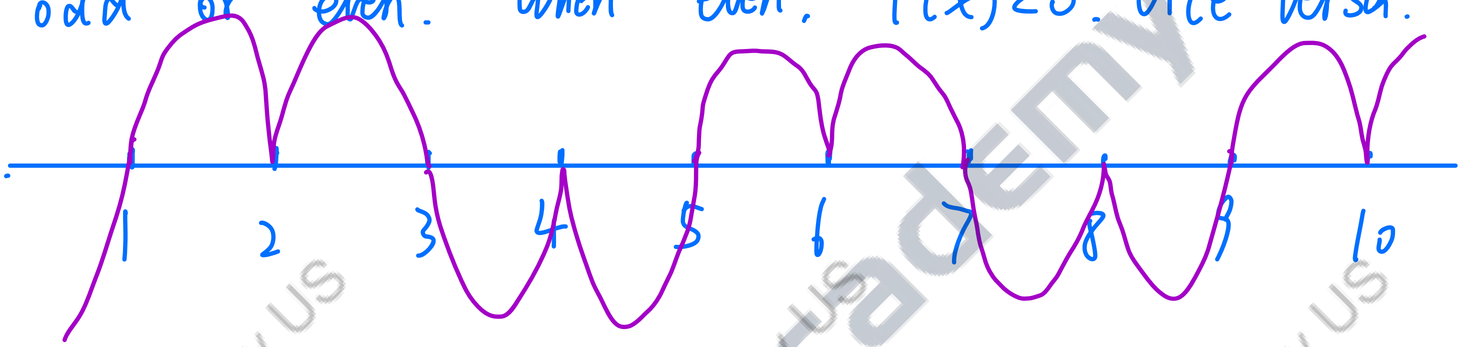
when $x > 10$, obviously $P(x) > 0$.

when $9 < x < 10$, because $(x-10)^{10} > 0$, $\therefore P(x) > 0$

when $8 < x < 9$, because $(x-9)^9 < 0$. $\therefore P(x) < 0$

when $7 < x < 8$, because $(x-8)^8 > 0$, $\therefore P(x) < 0$

In fact, when $n-1 < x < n$, whether $P(x)$ is positive or negative only depends on the number of positive odd numbers which are smaller than n is odd or even. when even, $P(x) < 0$. vice versa.



obviously, there are 6 intervals.

13. What is the area of the region in the coordinate plane defined by

$$||x|-1| + ||y|-1| \leq 1?$$

(A) 2 (B) 8 (C) 4 (D) 15 (E) 12

By examining the properties of absolute values, it's not difficult to notice that $|x|$ and $|y|$ actually only represent the scenario of the original graph of function being symmetrical about the y and x axes.

Therefore, we only need to consider

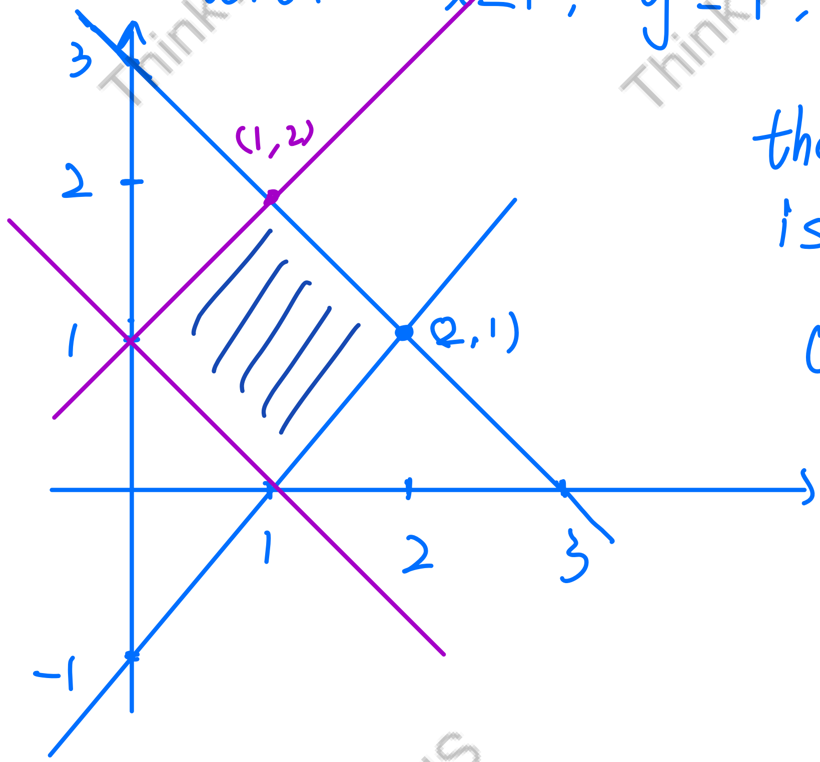
$$|x-1| + |y-1| \leq 1$$

when $x \geq 1$, $y \geq 1$, we have $x+y \leq 3$

when $x \geq 1$, $y \leq 1$, we have $x-y \leq 1$

when $x \le 1$, $y \ge 1$, we have $x - y \ge -1$

when $x \le 1$, $y \le 1$, we have $x + y \ge 1$



the area of the square is $\frac{1}{2} \times 2^2$ or $(\sqrt{2})^2$

(use diagonals) (use side length)

so for whole graph, is $2 \times 4 = 8$

14. How many ordered pairs of integers (m, n) satisfy the equation $m^2 + mn + n^2 = m^2n^2$?

(A) 7 **(B)** 1 **(C)** 3 **(D)** 6 **(E)** 5

$m^2 + 2mn + n^2 = m^2n^2 + mn \Rightarrow (m+n)^2 = mn(mn+1)$

$\because$ (mn) and $(mn+1)$ are coprime, so the only possible situation is one of them is 0.

1° $mn = 0$, by $m+n=0$. $(m,n) = (0,0)$

2° $mn+1=0$, by $m+n=0$, $(m,n) = (1,-1)$ or $(-1,1)$

$\therefore$ in total there are 3 pairs.

15. What is the least positive integer m such that $m \cdot 2! \cdot 3! \cdot 4! \cdot 5! \cdots 16!$ is a perfect square?

(A) 30 **(B)** 30,030 **(C)** 70 **(D)** 1430 **(E)** 1001

$2!\,3!\,4!\,5! \cdots 16! = 2 \times (3!)^2 \times 4 \times (5!)^2 \times 6 \times \cdots \times (15!)^2 \times 16$

$2 \times 4 \times 6 \times 8 \times 10 \times 12 \times 14 \times 16 = 2^{15} \cdot 3^2 \cdot 5 \cdot 7$

$\therefore\ m = 2 \times 5 \times 7 = 70$

16. Define an *upno* to be a positive integer of 2 or more digits where the digits are strictly increasing moving left to right. Similarly, define a *downno* to be a positive integer of 2 or more digits where the digits are strictly decreasing moving left to right. For instance, the number 258 is an upno and 8620 is a downno. Let U equal the total number of upnos and D equal the total number of downnos. What is $|U - D|$?

(A) 512 **(B)** 10 **(C)** 0 **(D)** 9 **(E)** 511

You can see that every downno $\overline{a_1 a_2 \cdots a_n}$ can corresponding to a upno $\overline{a_n \cdots a_2 a_1}$

UNLESS $a_n = 0$.

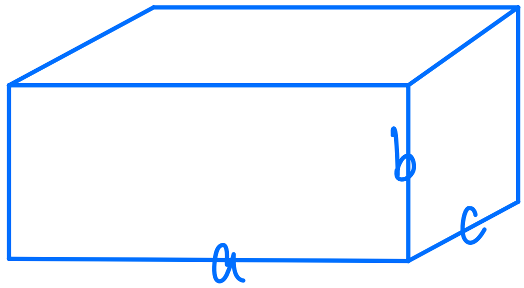
So in fact we are finding the number of downno ending with 0.

If we know 1, 2, 3 are in a downno, then the order is confirmed, it must be 321.

So in fact. we just need to know how many methods are there to choose more than 1 number from $1 \sim 9$. So $2^9 - 1 = 511$

17. A rectangular box $\mathcal{P}$ has distinct edge lengths a, b, and c. The sum of the lengths of all 12 edges of $\mathcal{P}$ is 13, the sum of the areas of all 6 faces of $\mathcal{P}$ is $\frac{11}{2}$, and the volume of $\mathcal{P}$ is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of $\mathcal{P}$?

(A) 2 **(B)** $\frac{3}{8}$ **(C)** $\frac{9}{8}$ **(D)** $\frac{9}{4}$ **(E)** $\frac{3}{2}$



$$\begin{cases} 4a+4b+4c=13 \\ 2ab+2bc+2ac=\frac{11}{2} \\ abc=\frac{1}{2} \end{cases}$$

$$\Rightarrow \begin{cases} a+b+c=\frac{13}{4} & ① \\ ab+bc+ac=\frac{11}{4} & ② \\ abc=\frac{1}{2} \end{cases}$$

$①^2-2\cdot②$:

$a^2+b^2+c^2=\left(\frac{13}{4}\right)^2-2\cdot\frac{11}{4}=\frac{81}{16}$. so $\sqrt{a^2+b^2+c^2}=\sqrt{\frac{81}{16}}=\frac{9}{4}$

18. Suppose that a, b, and c are positive integers such that

$$\frac{a}{14}+\frac{b}{15}=\frac{c}{210}.$$

Which of the following statements are necessarily true?

I. If $\gcd(a,14)=1$ or $\gcd(b,15)=1$ or both, then $\gcd(c,210)=1$.

II. If $\gcd(c,210)=1$, then $\gcd(a,14)=1$ or $\gcd(b,15)=1$ or both.

III. $\gcd(c,210)=1$ if and only if $\gcd(a,14)=1$ and $\gcd(b,15)=1$.

(A) I, II, and III **(B)** I only **(C)** I and II only **(D)** III only **(E)** II and III only

$\frac{a}{14}+\frac{b}{15}=\frac{15a+14b}{210}$ $\therefore c=15a+14b$.

(1) obviously, I is incorrect, it should be $\gcd(a,14)=1$ AND $\gcd(b,15)=1$, then $\gcd(c,210)=1$ (e.g. $a=1$, $b=15$, then $c=225$)

(2) suppose II is incorrect, that's means $\gcd(a,14)=p$, $\gcd(b,15)=q$, $p,q>1$

and p, q are integers, and $\gcd(c, 210) = 1$,

but $\frac{a}{14} = \frac{\frac{a}{p}}{\frac{14}{p}}$, $\frac{b}{15} = \frac{\frac{b}{q}}{\frac{15}{q}}$,

$$\frac{a}{14} + \frac{b}{15} = \frac{\frac{15a}{pq} + \frac{14b}{pq}}{\frac{210}{pq}} = \frac{c}{210}$$

$\therefore pq \mid c$, $pq \mid 210$, $\gcd(c, 210) \geq pq$ 矛

$\therefore$ II is correct.

(In fact, it is enough to choose E now)

(3) the "only if" part have been proved by (2), the "if" part have been proved by (1), $\therefore$ III is correct.

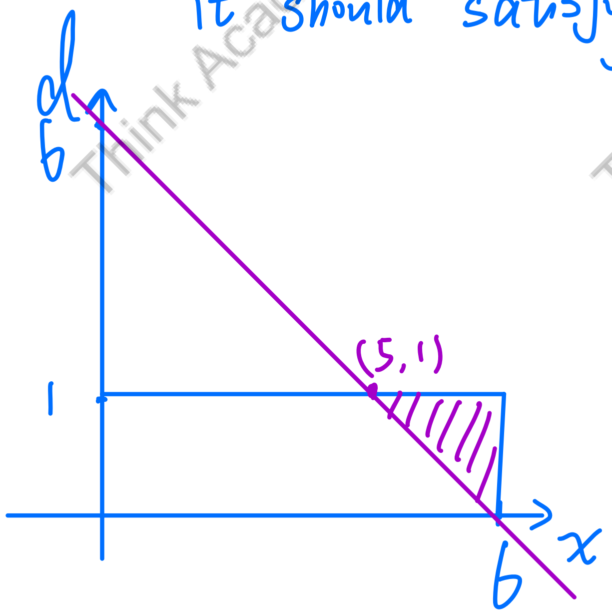
$\therefore$ Choose E.

19. Sonya the frog chooses a point uniformly at random lying within the square $[0, 6] \times [0, 6]$ in the coordinate plane and hops to that point. She then chooses a distance uniformly at random in the interval $[0, 1]$ and a direction uniformly at random from {north, east, south, west}. All her choices are independent. She now hops the chosen distance in the chosen direction. What is the probability that she lands outside the square?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{10}$ **(E)** $\frac{1}{9}$

In fact, the probability on each direction is equal.

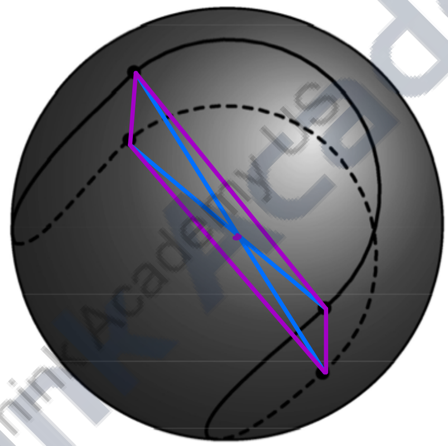
Suppose Sonya choose to jump east. For her initial x-coordinate and jump distance d, it should satisfy: $x+d>6$, which is to say



so the probability is

$$\frac{\frac{1\times 1}{2}}{6\times 1}=\frac{1}{12}$$

20. Four congruent semicircles are drawn on the surface of a sphere with radius 2, as shown, creating a closed curve that divides its surface into two congruent regions. The length of the curve is $\pi\sqrt{n}$. What is n ?



(A) 32 **(B)** 12 **(C)** 48 **(D)** 36 **(E)** 27

the purple part is a square with side length as the diameter of the semicircle and diagonal length as the diameter of the sphere which is $2\times 2=4$.

so the diameter of semicircle is $2\sqrt{2}$, radius is $\sqrt{2}$

The length of curve is $4\times\frac{2\cdot\pi\cdot\sqrt{2}}{2}=4\sqrt{2}\pi=\sqrt{32}\pi$

21. Each of 2023 balls is randomly placed into one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

(A) $\frac{2}{3}$ (B) $\frac{3}{10}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$ (E) $\frac{1}{4}$

Let a_n be the probability that each of the bins will contain an odd number of balls when there are $(2n+1)$ balls be randomly placed into one of 3 bins. $b_n = 1 - a_n$

$a_1 = \frac{{}_3P_3}{3^3} = \frac{2}{9}$. $b_1 = \frac{7}{9}$

when we have two more balls, to keep all odd, we can't put them into same bin. the probability is $\frac{1}{3}$, on the other hand, to change a 2-even-1-odd situation into 3-odd, we need to put them into 2 even-bins, probability is $\frac{2}{9}$

$a_{n+1} = \frac{1}{3}a_n + \frac{2}{9}b_n = \frac{1}{3}a_n + \frac{2}{9}(1-a_n)$

$= \frac{1}{9}a_n + \frac{2}{9}$, the fixed point of

$x = \frac{1}{9}x + \frac{2}{9}$ is $x = \frac{1}{4}$.

$\therefore a_{n+1} - \frac{1}{4} = \frac{1}{9}a_n + \frac{2}{9} - \frac{1}{4} = \frac{1}{9}a_n - \frac{1}{36}$

$$= \frac{1}{9}\left(a_n - \frac{1}{4}\right) = \cdots = \left(\frac{1}{9}\right)^n \left(a_1 - \frac{1}{4}\right)$$

$$\therefore a_{n+1} = \frac{1}{4} + \left(\frac{1}{9}\right)^n \cdot \frac{1}{36}$$

when n is very large, $\left(\frac{1}{9}\right)^n$ is so small that we can ignore. so answer is $\frac{1}{4}$

Solution 2: (fastest)

consider choose 3 numbers randomly, there are 4 situations to let the sum be odd, each of them is equal probability.

odd odd odd → only one of it meet our requirement, so the probability is $\frac{1}{4}$

odd even even

even odd even

even even odd

22. How many distinct values of x satisfy $\lfloor x \rfloor^2 - 3x + 2 = 0$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x ?

(A) an infinite number **(B)** 4 **(C)** 2 **(D)** 3 **(E)** 0

$$(x-1)^2 < \lfloor x \rfloor^2 = 3x-2 \le x^2$$

1° $x^2-3x+2 \geq 0 \Rightarrow (x-1)(x-2) \geq 0 \Rightarrow x \geq 2$ or $x \le 1$

2° $x^2-5x+3<0 \Rightarrow \frac{5-\sqrt{13}}{2} < x < \frac{5+\sqrt{13}}{2}$

$\therefore \lfloor x \rfloor = 0$ or 1 or 2 or 3 or 4

when $\lfloor x \rfloor = 0$, $x = \frac{\lfloor x \rfloor^2+2}{3} = \frac{2}{3}$

$\lfloor x \rfloor = 1$, $x = \frac{\lfloor x \rfloor^2+2}{3} = 1$

$\lfloor x \rfloor = 2$, $x = \frac{\lfloor x \rfloor^2+2}{3} = 2$

$\lfloor x \rfloor = 3$, $x = \frac{\lfloor x \rfloor^2+2}{3} = \frac{11}{3}$

$\lfloor x \rfloor = 4$, $x = \frac{\lfloor x \rfloor^2+2}{3} = \frac{17}{3}$, $\lfloor x \rfloor = 5 \Rightarrow \!\!\!\!/$

so there are 4 solutions.

23. An arithmetic sequence of positive integers has $n \geq 3$ terms, initial term a, and common difference $d > 1$. Carl wrote down all the terms in this sequence correctly except for one term, which was off by 1. The sum of the terms he wrote down was 222. What is $a + d + n$?

(A) 24 **(B)** 20 **(C)** 22 **(D)** 28 **(E)** 26

$$S = \frac{(a + a + (n-1)d)n}{2} = 221 \text{ or } 223$$

$221 = 13 \times 17$, 223 is a prime number

$\because d > 1$, $\therefore (n-1)d > n-1$, if $S = 223$,

then $n = 223 \Rightarrow \!\!\!\!/$ $\therefore S = 221$

$\because S > \dfrac{(2a+n-1)n}{2} > n^2$

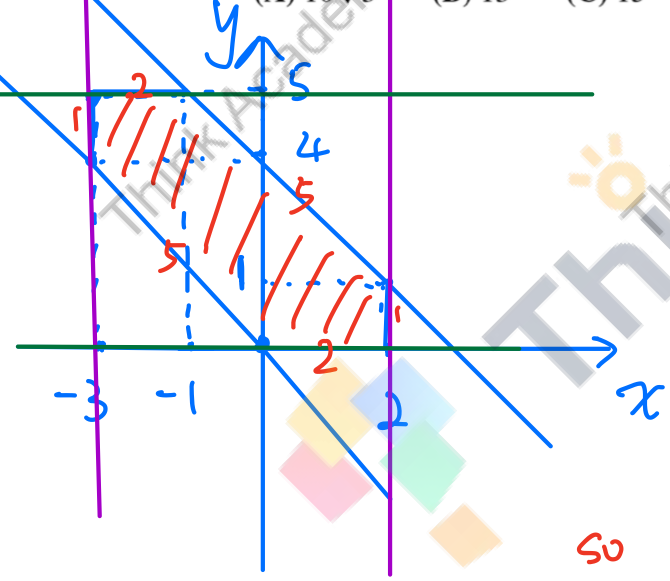
$\therefore n < \sqrt{221}$, $n \le 14$

$\therefore n = 13$, $\therefore a + 6d = 17$. $\because d > 1$

$\therefore d = 2$, $a = 5$. $n + d + a = 20$

24. What is the length of the boundary of the region in the xy plane consisting of points of the form $(\underbrace{2u-3w}_{x}, \underbrace{v+4w}_{y})$ where $0 \le u \le 1$, $0 \le v \le 1$, and $0 \le w \le 1$?

(A) $10\sqrt{3}$ **(B)** 13 **(C)** 15 **(D)** 18 **(E)** 16



$4x + 3y = 8u + 3v \in [0, 11]$

$x = 2u - 3w \in [-3, 2]$

$y = u + 4w \in [0, 5]$

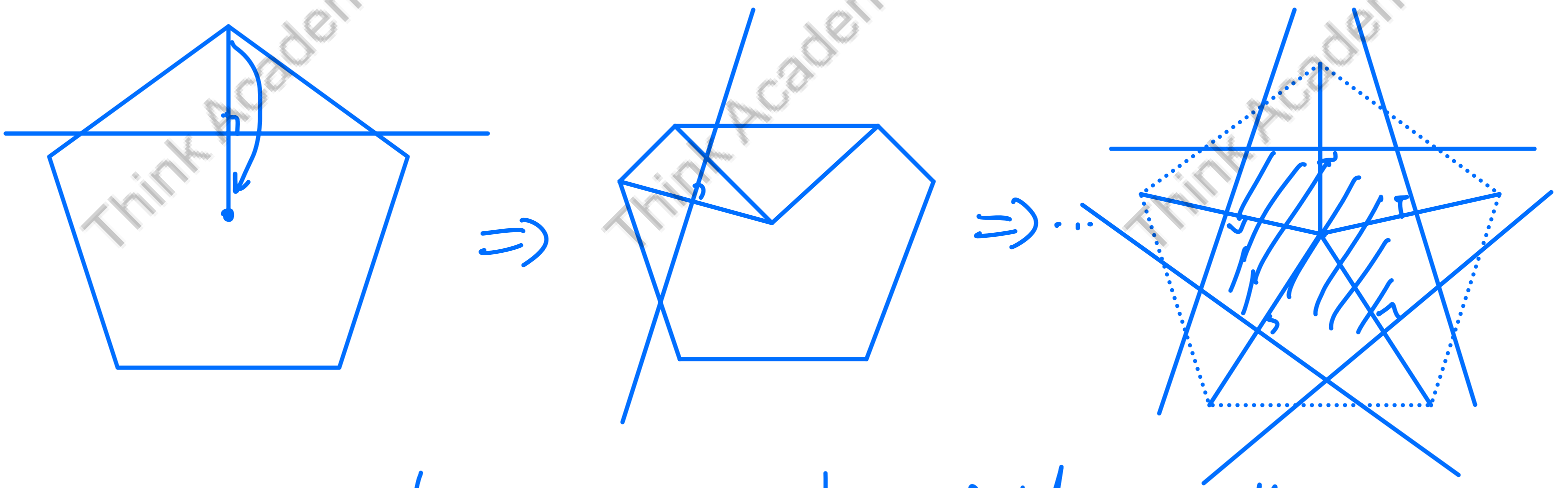
So the red part are the region

so the length of boundary is

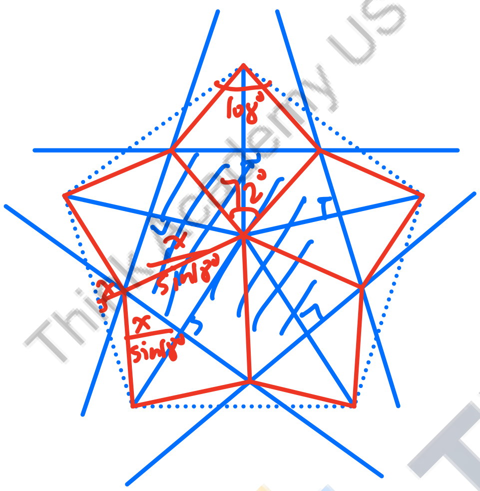
$2 \times (1 + 2 + 5) = 16$

25. A regular pentagon with area $1 + \sqrt{5}$ is printed on paper and cut out. All five vertices are folded to the center of the pentagon, creating a smaller pentagon. What is the area of the new pentagon?

(A) $4 - \sqrt{5}$ **(B)** $\sqrt{5} - 1$ **(C)** $8 - 3\sqrt{5}$ **(D)** $\dfrac{1+\sqrt{5}}{2}$ **(E)** $\dfrac{2+\sqrt{5}}{3}$



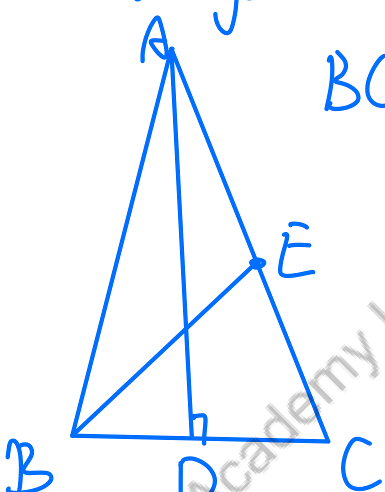
we get the new one by folding all the perpendicular bisectors of the segments between vertices and center.



the 5 red part are rhombuses so if you know about sin 18° (golden ratio) you can solve about it directly.

$$\text{area} = (\sqrt{5}+1)\cdot\frac{\frac{1}{\sin 18^\circ}}{\frac{1}{\sin 18^\circ}+1}\cdot\frac{1}{2} = \sqrt{5}-1$$

if you don't know, here's method to use similar triangle to get it: isosceles $\triangle ABC$ satisfy: $\angle BAC = 36^\circ$, $BC=1$, $\angle B = \angle C = 72^\circ$. make $AD \perp BC$ at D. draw the angle bisector of $\angle B$ intersect AC at point E. suppose $AB = x$.



by angle bisector theorem, $\frac{EC}{AE} = \frac{BC}{AB} = \frac{1}{x}$

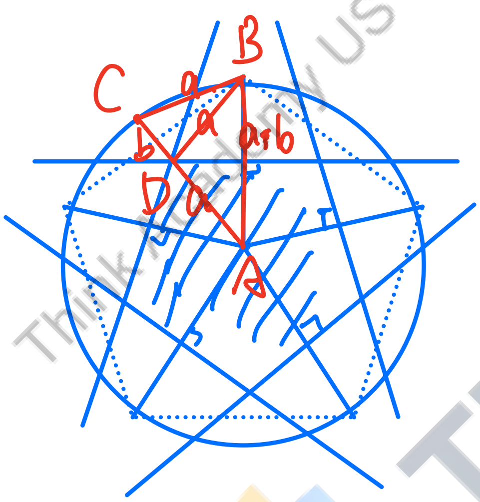
$\because EC + AE = AC = x$, $\therefore EC = \dfrac{x}{x+1}$. $\because \triangle BCE \sim \triangle ABC$

$\therefore \dfrac{CE}{BC} = \dfrac{BC}{AB}$, $\therefore \dfrac{\frac{x}{x+1}}{1} = \dfrac{1}{x} \Rightarrow x^2 - x - 1 = 0$. $\because x > 0$

$\therefore x = \dfrac{1+\sqrt{5}}{2}$, $\therefore \sin 18^\circ = \dfrac{DC}{AC} = \dfrac{\frac{1}{2}}{\frac{1+\sqrt{5}}{2}} = \dfrac{1}{\sqrt{5}+1}$

$= \dfrac{\sqrt{5}-1}{4}$.

Solution 2:



$\triangle BCD \sim \triangle ABC$.

$\Rightarrow \dfrac{b}{a} = \dfrac{a}{a+b} \Rightarrow \dfrac{a}{b} + 1 = \left(\dfrac{a}{b}\right)^2$

$\therefore \dfrac{a}{b} = \dfrac{\sqrt{5}+1}{2}$ $\left(\because \dfrac{a}{b} > 0\right)$

$\therefore \dfrac{A_{\text{smaller pentagon}}}{A_{\text{Larger pentagon}}} = \left(\dfrac{a}{a+b}\right)^2 = \left(\dfrac{b}{a}\right)^2$

$= \dfrac{3-\sqrt{5}}{2}$

$\therefore A_{\text{smaller pentagon}} = (\sqrt{5}+1)\dfrac{(3-\sqrt{5})}{2} = \sqrt{5}-1$