Official Solutions

MAA American Mathematics Competitions

26th Annual

AMC 10 B

Tuesday, November 12, 2024

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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Questions and comments about this competition should be sent to

amcinfo@maa.org

or

MAA American Mathematics Competitions
P.O. Box 471
Annapolis Junction, MD 20701.

The problems and solutions for this AMC 10 B were prepared by the
MAA AMC 10/12 Editorial Board under the direction of
Gary Gordon and Carl Yerger, co-Editors-in-Chief.

1. In a long line of people arranged left to right, the 1013th person from the left is also the 1010th person from the right. How many people are in the line?

(A) 2021 **(B)** 2022 **(C)** 2023 **(D)** 2024 **(E)** 2025

Answer (B): There are 1012 people to the left of the specified person and 1009 people to the right of that person. There are therefore $1012 + 1 + 1009 = 2022$ people in the line.

2. What is $10! - 7! \cdot 6!$?

(A) -120 **(B)** 0 **(C)** 120 **(D)** 600 **(E)** 720

Answer (B): Note that

$$10! - 7! \cdot 6! = 7! \cdot (10 \cdot 9 \cdot 8 - 6!) = 7! \cdot (720 - 720) = 0.$$

3. For how many integer values of x is $|2x| \leq 7\pi$?

(A) 16 **(B)** 17 **(C)** 19 **(D)** 20 **(E)** 21

Answer (E): From $3 < \pi < 3.142$, it follows that

$$21 = 7 \cdot 3 < 7\pi < 7 \cdot 3.142 = 21.994 < 22.$$

Because x is an integer, the values for $2x$ that make the inequality true are $-20, -18, -16, \ldots, -2$, $0, 2, \ldots, 18$, and 20. Each of these corresponds to a unique value of x. There are 21 such values.

4. Balls numbered $1, 2, 3, \ldots$ are deposited in 5 bins, labeled A, B, C, D, and E, using the following procedure. Ball 1 is deposited in bin A, and balls 2 and 3 are deposited in bin B. The next 3 balls are deposited in bin C, the next 4 in bin D, and so on, cycling back to bin A after balls are deposited in bin E. (For example, balls numbered $22, 23, \ldots, 28$ are deposited in bin B at step 7 of this process.) In which bin is ball 2024 deposited?

(A) A **(B)** B **(C)** C **(D)** D **(E)** E

Answer (D): After n steps, a total of $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ balls have been deposited. In particular, after step $n = 63$, a total of $\frac{1}{2} \cdot 63 \cdot 64 = 2016$ balls have been deposited. The next batch of 64 balls will include ball 2024. Because 64 has remainder 4 when divided by 5, ball 2024 is deposited in bin D.

5. In the following expression, Melanie changed some of the plus signs to minus signs:

$$1 + 3 + 5 + 7 + \cdots + 97 + 99.$$

When the new expression was evaluated, it was negative. What is the least number of plus signs that Melanie could have changed to minus signs?

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

Answer (B): To minimize the number of minus signs needed to make the expression negative, minus signs should be chosen for all of the largest numbers. Hence the first k numbers of the expression will stay positive and the last $50-k$ will be made negative for the greatest value of k that gives a negative value.

Recall that $1+3+5+\cdots+(2n-1)=n^2$; that is, the sum of the first n odd positive integers is equal to n^2. The expression in the problem statement is the sum of the first 50 odd positive integers, so it equals $50^2=2500$. Hence k^2 must be strictly less than $\frac{2500}{2}=1250$. Because $35^2=1225$ and $36^2=1296$, at least 15 plus signs must be switched to minus signs for the expression to evaluate to a negative value. Indeed,

$$1+3+5+\cdots+69-71-73-75-\cdots-99=1225-(2500-1225)=-50.$$

6. A rectangle has integer length sides and an area of 2024. What is the least possible perimeter of the rectangle?

   (A) 160 **(B)** 180 **(C)** 222 **(D)** 228 **(E)** 390

   Answer (B): Note that $2024=44\cdot 46$. A 44×46 rectangle will have perimeter $2(44+46)=180$. It is straightforward to check the other possible dimensions to show that this gives the rectangle with the least possible perimeter:

   - 23×88 gives a perimeter of $2(23+88)=222$.
   - 22×92 gives a perimeter of $2(22+92)=228$.
   - 11×184 gives a perimeter of $2(11+184)=390$.
   - If one of the dimensions is $1,2,4,$ or 8, then the other dimension is greater than 200, yielding rectangles with greater perimeters.

   Thus the least possible perimeter is 180.

7. What is the remainder when $7^{2024}+7^{2025}+7^{2026}$ is divided by 19 ?

   (A) 0 **(B)** 1 **(C)** 7 **(D)** 11 **(E)** 18

   Answer (A): The quantity in question is seen to be a multiple of 19 as follows:

   $$7^{2024}+7^{2025}+7^{2026}=7^{2024}\left(1+7+7^2\right)=7^{2024}\cdot 57=7^{2024}\cdot 3\cdot 19.$$

   Therefore the remainder when it is divided by 19 is 0.

   OR

   Working modulo 19, $7^0=1$, $7^1=7\cdot 1=7$, $7^2=7\cdot 7=49\equiv 11$, and $7^3\equiv 7\cdot 11=77\equiv 1$. Therefore the remainders when dividing successive nonnegative powers of 7 by 19 repeat with period 3. The sum of any three consecutive remainders is therefore $1+7+11=19$, so the remainder when the given sum is divided by 19 is 0.

8. Let N be the product of all the positive integer divisors of 42. What is the units digit of N ?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Answer (D): The prime factorization of 42 is $2 \cdot 3 \cdot 7$. A divisor of 42 is determined by choosing whether or not to include each prime as a factor of the divisor, so there are $2^3 = 8$ divisors. Furthermore, the divisors can be formed into 4 pairs each of whose product is 42: $1 \cdot 42$, $2 \cdot 21$, $3 \cdot 14$, and $6 \cdot 7$. Therefore $N = 42^4$. The value of the requested units digit of N is the same as the units digit of 2^4, which is 6.

9. Real numbers a, b, and c have arithmetic mean 0. The arithmetic mean of a^2, b^2, and c^2 is 10. What is the arithmetic mean of ab, ac, and bc ?

(A) -5 **(B)** $-\dfrac{10}{3}$ **(C)** $-\dfrac{10}{9}$ **(D)** 0 **(E)** $\dfrac{10}{9}$

Answer (A): The given information implies that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 30$. Then

$$0 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 30 + 2(ab + ac + bc).$$

Therefore $2(ab + ac + bc) = -30$ and the requested arithmetic mean is $\frac{ab+ac+bc}{3} = \frac{-15}{3} = -5$.

OR

Consider the system of equations implied by the conditions of the problem,

$$\begin{aligned} a + b + c &= 0 \\ a^2 + b^2 + c^2 &= 30, \end{aligned}$$

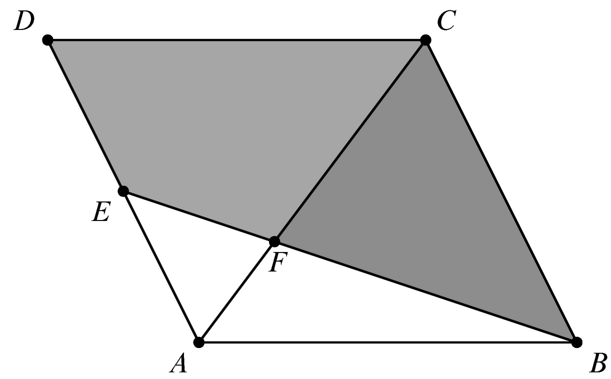
and suppose that $a = 0$. Then $b + c = 0$, so $b = -c$, and substituting into the second equation gives $2b^2 = 30$, from which $b = \pm\sqrt{15}$ and $c = \mp\sqrt{15}$. If one assumes that the requested arithmetic mean is determined by the given information, independent of the value of a, then

$$\frac{ab + ac + bc}{3} = \frac{0 + 0 - \sqrt{15} \cdot \sqrt{15}}{3} = -5.$$

10. Quadrilateral $ABCD$ is a parallelogram, and E is the midpoint of the side $\overline{AD}$. Let F be the intersection of lines EB and AC. What is the ratio of the area of quadrilateral $CDEF$ to the area of $\triangle CFB$?

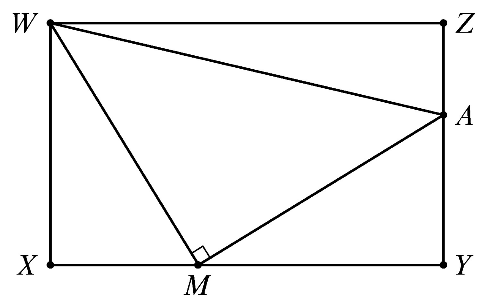
(A) $5:4$ **(B)** $4:3$ **(C)** $3:2$ **(D)** $5:3$ **(E)** $2:1$

Answer (A): Triangles $\triangle AFE$ and $\triangle CFB$ are similar by Angle-Angle. The ratio of corresponding sides is $1:2$, so the ratio of their areas is $1:4$. Furthermore, consider $\triangle AFE$ and $\triangle AFB$. Because $FE:FB = 1:2$ and the heights of the two triangles corresponding to bases $\overline{FE}$ and $\overline{FB}$ are the same, their areas are in a $1:2$ ratio. Finally, observe that the area of $\triangle ABE$ is $\frac{1}{4}$ the area of parallelogram $ABCD$. Therefore, using the area of $\triangle AFE$ as unit, the area of $\triangle CFB$ is 4, the area of $\triangle AFB$ is 2, the area of quadrilateral $ABCD$ is $4 \cdot (1 + 2) = 12$, and the area of quadrilateral $CDEF$ is $12 - (1 + 4 + 2) = 5$. The requested ratio of the areas of quadrilateral $CDEF$ and $\triangle CFB$ is $5:4$.



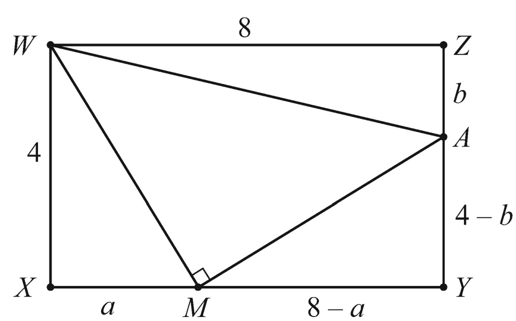
Note: The answer can also be obtained by assuming that the parallelogram is a square with vertices $(0,0)$, $(12,0)$, $(12,12)$, and $(0,12)$.

11. In the figure below $WXYZ$ is a rectangle with $WX = 4$ and $WZ = 8$. Point M lies on $\overline{XY}$, point A lies on $\overline{YZ}$, and $\angle WMA$ is a right angle. The areas of triangles $\triangle WXM$ and $\triangle WAZ$ are equal. What is the area of $\triangle WMA$?



(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

Answer (C): Label the diagram as shown, where $MX = a$ and $ZA = b$.



The Pythagorean Theorem on $\triangle WMA$ gives $WM^2 + MA^2 = WA^2$, which implies that

$$4^2 + a^2 + (8-a)^2 + (4-b)^2 = 8^2 + b^2.$$

Expanding and simplifying yields $a^2 - 8a - 4b + 16 = 0$. Because the areas of triangles $\triangle WXM$ and $\triangle WAZ$ are equal, $\frac{1}{2} \cdot 4a = \frac{1}{2} \cdot 8b$, so $a = 2b$. Substituting into the previous equation and factoring gives $4(b-1)(b-4)=0$. Therefore $b=1$ or $b=4$. But $b=4$ would require $A=Y=M$, and $\angle WMA$ would not exist, so it must be that $b=1$ and $a=2$. The area of $\triangle WMA$ can be found by subtracting the three other triangle areas from the area of the rectangle:

$$8\cdot 4-\frac{1}{2}\cdot 4\cdot 2-\frac{1}{2}\cdot 6\cdot 3-\frac{1}{2}\cdot 8\cdot 1=32-4-9-4=15.$$

12. A group of 100 students from different countries meet at a mathematics competition. Each student speaks the same number of languages, and, for every pair of students A and B, student A speaks some language that student B does not speak, and student B speaks some language that student A does not speak. What is the least possible total number of languages spoken by all the students?

(A) 9 **(B)** 10 **(C)** 12 **(D)** 51 **(E)** 100

Answer (A): Suppose the languages spoken are labeled $L_1, L_2, L_3, \ldots, L_n$. Note that the collection of all subsets of $\{L_1, L_2, L_3, \ldots, L_n\}$ of size r will satisfy the conditions in the problem for any r in the range $1 \le r < n$. For a given n, there are $\binom{n}{r}$ subsets, and $\binom{n}{r}$ is maximized when $r=\left\lfloor \frac{n}{2}\right\rfloor$ or $r=\left\lceil \frac{n}{2}\right\rceil$.

If $n=8$ and $r=4$, the number of distinct subsets is $\binom{8}{4}=70<100$, so $n>8$. But $\binom{9}{4}=\binom{9}{5}=126>100$, so $n=9$ is the least possible total number of languages spoken by all the students.

Note: The restriction on having the students speak the same number of languages is not needed. A collection of subsets of a given set with n elements satisfying the condition that no member of the collection is a subset of another is called an *antichain*. Sperner's Theorem asserts that the largest antichain occurs when the subsets are all of size r, where $r=\left\lfloor \frac{n}{2}\right\rfloor$ or $\left\lceil \frac{n}{2}\right\rceil$.

13. Positive integers x and y satisfy the equation $\sqrt{x}+\sqrt{y}=\sqrt{1183}$. What is the minimum possible value of $x+y$?

(A) 585 **(B)** 595 **(C)** 623 **(D)** 700 **(E)** 791

Answer (B): Observe that $1183=13^2\cdot 7$, so $\sqrt{1183}=13\sqrt{7}$. Because $\sqrt{x}+\sqrt{y}=13\sqrt{7}$, it follows that $\sqrt{x}$ and $\sqrt{y}$ must be of the form $a\sqrt{7}$ and $b\sqrt{7}$, respectively, where a and b are positive integers and $a+b=13$. Then $\sqrt{x}=\sqrt{7a^2}$ and $\sqrt{y}=\sqrt{7b^2}$, so $x=7a^2$ and $y=7b^2$. Substituting gives

$$x+y=7\left(a^2+b^2\right)=7\left(a^2+(13-a)^2\right)=14a^2-182a+1183.$$

The minimum value of the quadratic polynomial occurs at $a=\frac{182}{2\cdot 14}=6.5$. Because a and b must be positive integers, without loss of generality (by symmetry), choose $a=6$ and $b=7$. The minimum possible value of $x+y$ is

$$x+y=7\cdot\left(6^2+7^2\right)=7\cdot(36+49)=7\cdot 85=595.$$

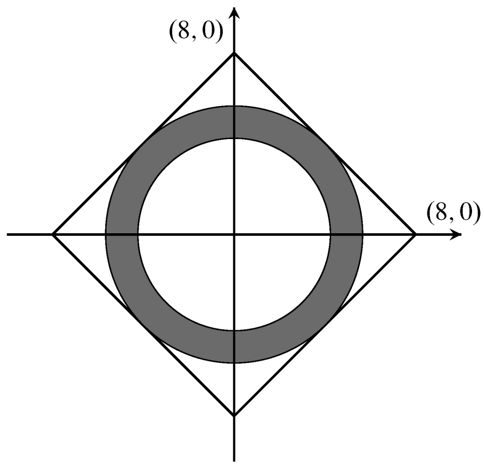
14. A dartboard is the region B in the coordinate plane consisting of points (x, y) such that $|x|+|y|\le 8$. A target T is the region where $\left(x^2+y^2-25\right)^2\le 49$. A dart is thrown and lands at a random point in B. The probability that the dart lands in T can be expressed as $\frac{m}{n}\cdot\pi$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 39 **(B)** 71 **(C)** 73 **(D)** 75 **(E)** 135

Answer (B): The region B is a square with intercepts $(\pm 8, 0)$ and $(0, \pm 8)$. The area of this square is $\left(8\sqrt{2}\right)^2 = 128$. Taking square roots shows that region T is the set of points that satisfy

$$25 - 7 \le x^2 + y^2 \le 25 + 7,$$

which is an annulus (ring) with inner radius $\sqrt{18}$ and outer radius $\sqrt{32}$. Its area is $32\pi - 18\pi = 14\pi$. Note that T is internally tangent to B at the four points $(\pm 4, \pm 4)$. The required probability is the ratio of the area of T to the area of B, namely $\frac{14\pi}{128} = \frac{7}{64}\pi$. The requested sum is $7 + 64 = 71$.



15. A list of 9 real numbers consists of 1, 2.2, 3.2, 5.2, 6.2, and 7, as well as x, y, and z with $x \le y \le z$. The range of the list is 7, and the mean and the median are both positive integers. How many ordered triples (x, y, z) are possible?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** infinitely many

Answer (C): Because the range is 7, the values of x, y, and z are in the interval $[0, 8]$. Because the median is an integer, it is one of x, y, or z and is either 3, 4, 5, or 6. The sum of the list is $s = 24.8 + x + y + z$, which is between $24.8 + 0 + 0 + 3 = 27.8$ and $24.8 + 6 + 8 + 8 = 46.8$. Because the mean is an integer, s is an integer multiple of 9, so $s = 36$ or 45, and $x + y + z = 11.2$ or 20.2.

- If the median is 3, then $x \le y \le z = 3$, so $x + y + z \le 9 < 11.2$. Therefore this case cannot occur.
- If the median is 4, then $x \le y = 4 \le z$, so $x + z = 7.2$, and the list spans one of the intervals $[0, 7]$, $[1, 8]$, or $[x, z]$. If $x = 0$, then $z = 7.2 > 7$, and if $z = 8$, then $x = -0.8 < 0$. Therefore these cases cannot occur. Otherwise $z - x = 7$, giving $(x, y, z) = (0.1, 4, 7.1)$.
- If the median is 5, then $x \le y = 5 \le z$, so $x + z = 6.2$. In order to have a range of 7, x must be 0, and $(x, y, z) = (0, 5, 6.2)$.
- If the median is 6, then $x = 6 \le y \le z$, so $y + z = 14.2$. In order to have a range of 7, z must be 8, and $(x, y, z) = (6, 6.2, 8)$.

Thus there are 3 possible ordered triples (x, y, z).

16. Jerry likes to play with numbers. One day, he wrote all the integers from 1 to 2024 on the whiteboard. Then he repeatedly chose four numbers on the whiteboard, erased them, and replaced them by either their sum or their product. (For example, Jerry's first step might have been to erase 1, 2, 3, and 5, and then write either 11, their sum, or 30, their product, on the whiteboard.) After repeatedly performing this operation, Jerry noticed that all the remaining numbers on the whiteboard were odd. What is the maximum possible number of integers on the whiteboard at that time?

(A) 1010 **(B)** 1011 **(C)** 1012 **(D)** 1013 **(E)** 1014

Answer (A): Each time this operation was performed, the number of even integers on the whiteboard was reduced by at most 3. There were $\frac{2024}{2} = 1012$ even integers on the whiteboard initially. Because $\frac{1011}{3} = 337$ and $\frac{1014}{3} = 338$, Jerry needed at least 338 operations to eliminate all of them. After 338 operations there were $2024 - (338 \cdot 3) = 1010$ numbers on the whiteboard.

To see how all the even numbers could have been eliminated using these operations, suppose Jerry chose $6n + 1$, $6n + 2$, $6n + 4$, and $6n + 6$ and replaced them by their sum, for $0 \le n \le 336$. For the final operation, Jerry could have erased 3, 5, 9, and 2024 and replaced them with their sum. Then after these 338 operations, all 1010 numbers on the whiteboard were odd.

17. In a race among 5 snails, there is at most one tie, but that tie can involve any number of snails. For example, the result of the race might be that Dazzler is first; Abby, Cyrus, and Elroy are tied for second; and Bruna is fifth. How many different results of the race are possible?

(A) 180 **(B)** 361 **(C)** 420 **(D)** 431 **(E)** 720

Answer (D): If there are no ties, then there are 5! possible race results. Suppose k of the 5 snails are tied, where $2 \le k \le 5$. There are $\binom{5}{k}$ ways to choose the snails that are tied, and then considering those snails as a group, there are $6 - k$ entrants and therefore $(6 - k)!$ orders of finish. The number of possible results is thus

$$5! + \binom{5}{2} \cdot 4! + \binom{5}{3} \cdot 3! + \binom{5}{4} \cdot 2! + \binom{5}{5} \cdot 1! = 120 + 240 + 60 + 10 + 1 = 431.$$

18. How many different remainders can result when the 100th power of an integer is divided by 125 ?

(A) 1 **(B)** 2 **(C)** 5 **(D)** 25 **(E)** 125

Answer (B): Write $N = 5k + r$ for $r = 0$, 1, 2, 3, or 4. If $r = 0$, then $N = 5k$ and N^{100} is divisible by 125, so the remainder is 0. If $r = 1$, 2, 3, or 4, then $N^2 = 25k^2 + 10rk + r^2 = 5m \pm 1$ for some integer m. Now use the Binomial Theorem:

$$\begin{aligned} N^{100} = (N^2)^{50} = (5m \pm 1)^{50} = (5m)^{50} \pm 50(5m)^{49} + \binom{50}{2}(5m)^{48} \pm \cdots \\ \pm \binom{50}{47}(5m)^3 + \binom{50}{48}(5m)^2 \pm 50(5m) + 1. \end{aligned}$$

All the terms except the final term have at least 3 factors of 5, so N^{100} has remainder 1 upon division by 125. Therefore there are only 2 possible remainders: 0 and 1.

OR

Let $\phi(n)$ be the number of positive integers less than n that are relatively prime to n; this is Euler's totient function. Then $\phi(125) = 5^3 - 5^2 = 100$. By Euler's Totient Theorem, if a is not a multiple of 5, then $a^{100} \equiv 1 \pmod{125}$. If a is a multiple of 5, then $a^{100} \equiv 0 \pmod{125}$. Therefore there are only 2 possible remainders: 0 and 1.

19. In the following table, each question mark is to be replaced by "Possible" or "Not Possible" to indicate whether a nonvertical line with the given slope can contain the given number of lattice points (points both of whose coordinates are integers). How many of the 12 entries will be "Possible"?

	zero	**exactly one**	**exactly two**	**more than two**
zero slope	**?**	**?**	**?**	**?**
nonzero rational slope	**?**	**?**	**?**	**?**
irrational slope	**?**	**?**	**?**	**?**

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 9

Answer (C): If the slope is 0, then the line is horizontal and its equation is $y = b$ for some real number b. If b is an integer, then the line will contain infinitely many lattice points, and if b is not an integer, then it will contain no lattice points. Therefore exactly two of the entries in that row of the table are "Possible".

Next suppose that the equation of the line is $y = mx + b$, where the slope m is a nonzero rational number, say $m = \frac{p}{q}$ for integers p and q with $q \neq 0$. If the line contains a lattice point (r, s), then it also contains the lattice points $(r + q, s + p)$, $(r + 2q, s + 2p)$, $(r + 3q, s + 3p)$, and so on. Therefore the fourth entry in that row of the table is "Possible" and the second and third entries are "Not Possible". To see that the line may contain no lattice points, let b be irrational. Then $(0, b)$ is a point on the line, but if (r, s) were a lattice point on the line, then

$$m = \frac{s - b}{r - 0}$$

would be an irrational number, a contradiction. Thus the first entry in the "nonzero rational slope" row of the table is "Possible". (This case actually includes the case of zero slope.)

Finally suppose that the equation of the line is $y = mx + b$, where the slope m is an irrational number. The line could certainly contain exactly one lattice point; for example, the equation of the line could be $y = \sqrt{2}x$ and the only lattice point on the line is $(0,0)$. It could also contain no lattice points; for example, its equation could be $y = \sqrt{2}x + \frac{1}{2}$. But if a nonvertical line contains two or more lattice points, say (r,s) and (t,u) with $r \neq t$, then its slope, $\frac{s-u}{r-t}$, is rational. Therefore the first and second entries in the bottom row of the table are "Possible" and the third and fourth entries are "Not Possible".

In all, 6 of the 12 entries are "Possible" (indicated by **P** in the table below), and 6 are "Not Possible" (indicated by **NP**).

	zero	**exactly one**	**exactly two**	**more than two**
zero slope	**P**	**NP**	**NP**	**P**
nonzero rational slope	**P**	**NP**	**NP**	**P**
irrational slope	**P**	**P**	**NP**	**NP**

20. Three different pairs of shoes are placed in a row so that no left shoe is next to a right shoe from a different pair. In how many ways can these six shoes be lined up?

(A) 60 **(B)** 72 **(C)** 90 **(D)** 108 **(E)** 120

Answer (A): There are $\binom{6}{3} = 20$ arrangements of the letters LLLRRR representing the positions of 3 left shoes and 3 right shoes in the row of 6 shoes. Of the 20, any sequence containing LRL or RLR will violate the condition given in the problem. There are 8 arrangements that avoid these two sequences. Call a pair of shoes *matched* if the 2 shoes in the pair are next to each other. There are three sets of possibilities.

- LLLRRR and RRRLLL: In these two cases, only one pair of shoes is matched. There are 3 choices for that pair, and there are $2 \cdot 2 = 4$ ways to place the other 4 shoes for a total of $6 \cdot 4 = 24$ arrangements for this case.
- LRRRLL, LLRRRL, RLLLRR, and RRLLLR: In each of these four cases, there are $3! = 6$ ways to place the left shoes, but then there is a unique way to place the right shoes, for a total of $6 \cdot 4 = 24$ arrangements in this case.
- LRRLLR and RLLRRL: In these two cases, all three pairs of shoes are matched, so, in each case, there are $3! = 6$ ways to place the shoes. This gives a total of $6 \cdot 2 = 12$ arrangements.

Thus there are $24 + 24 + 12 = 60$ arrangements satisfying the conditions of the problem.

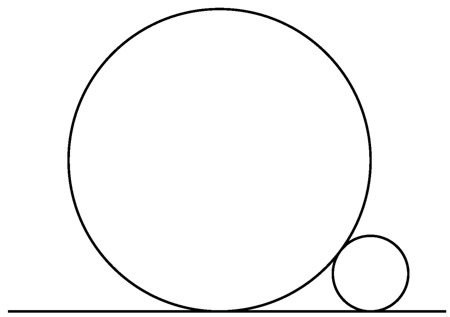
OR

Label the shoes L_i and R_i for $i = 1, 2, 3$. By symmetry it suffices to count the number of arrangements with L_1 first in line and multiply by 6. There are 2 choices for a left shoe coming next, say

L_2, after which the only continuations are $L_3R_3R_jR_{3-j}$ for $j = 1$ or 2, or $R_2R_1R_3L_3$; this gives $2 \cdot (2+1) = 6$ arrangements. Otherwise R_1 comes second, followed by either $R_kR_{5-k}L_{5-k}L_k$ or $R_kL_kL_{5-k}R_{5-k}$ for $k = 2$ or 3, another $2+2 = 4$ arrangements. This gives a total of 10, so there are $6 \cdot 10 = 60$ ways to line up the six shoes.

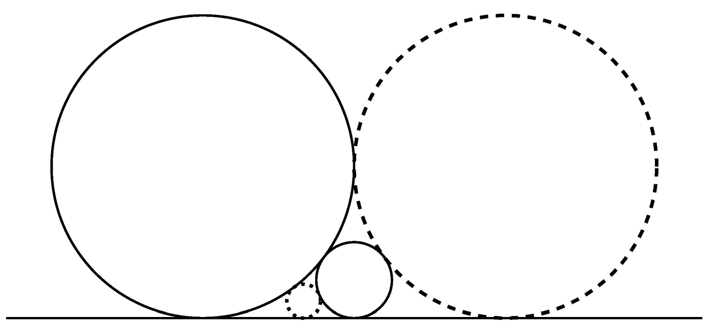
Note: Generalized to n pairs of shoes, the sequence of answers to this question starts $2, 8, 60, 816,$ 17520. This is sequence A096121 in the On-Line Encyclopedia of Integer Sequences, which is listed as counting the number of paths of a rook on a $2 \times n$ board, where the rook must land on each square exactly once. The sequence satisfies the recurrence $a_{n+1} = n(n+1)(a_n + a_{n-1})$.

21. Two straight pipes (circular cylinders), with radii 1 and $\frac{1}{4}$, lie parallel and in contact on a flat floor. The figure below shows a head-on view. What is the sum of the possible radii of a third parallel pipe lying on the same floor and in contact with both?

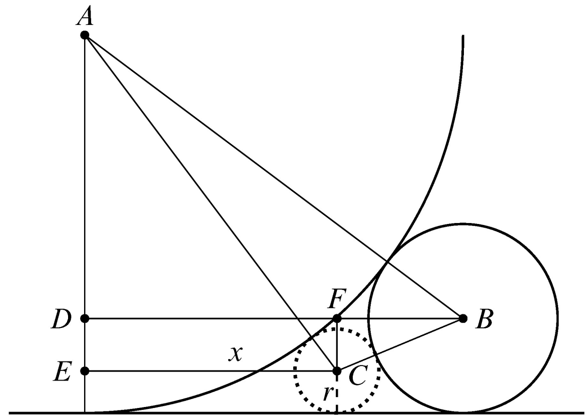


(A) $\frac{1}{9}$ **(B)** 1 **(C)** $\frac{10}{9}$ **(D)** $\frac{11}{9}$ **(E)** $\frac{19}{9}$

Answer (C): There are two possible positions for the third pipe—either nestled in the gap between the pipes or outside. See the figure below.



Consider the blown-up figure below. In this diagram, A is the center of the circle of radius 1, B is the center of the circle of radius $\frac{1}{4}$, and C is the center of the third circle nestled in the gap. The horizontal lines through B and C intersect the vertical line through A at D and E, respectively, and F is the foot of the perpendicular from C to $\overline{BD}$.



Because $AB = 1 + \frac{1}{4} = \frac{5}{4}$ and $AD = 1 - \frac{1}{4} = \frac{3}{4}$, it follows that $\triangle ADB$ is a $3-4-5$ right triangle scaled down by a factor of 4, so $BD = \frac{4}{4} = 1$. Thus the vertical line through B is tangent to the given circle of radius 1. Then by symmetry, the radius of the larger of the two dashed circles tangent to both given circles has radius 1.

It remains to compute the radius r of the smaller dashed tangent circle. Let $x = CE = DF$. The Pythagorean Theorem in $\triangle AEC$ gives $x^2 + (1-r)^2 = (1+r)^2$, which simplifies to $x^2 = 4r$. The Pythagorean Theorem in $\triangle BFC$ gives

$$(1-x)^2 + \left(1 - \frac{3}{4} - r\right)^2 = \left(\frac{1}{4} + r\right)^2,$$

which simplifies to $(1-x)^2 = r$. Combining these equations gives $x^2 = 4(1-x)^2$, which is equivalent to $3x^2 - 8x + 4 = 0$ and $(3x-2)(x-2) = 0$. Because $x < 1$, the relevant solution is $x = \frac{2}{3}$, and the radius of the smaller circle is $\frac{1}{4} \cdot \left(\frac{2}{3}\right)^2 = \frac{1}{9}$.

The requested sum of possible radii is $1 + \frac{1}{9} = \frac{10}{9}$.

22. A group of 16 people will be partitioned into 4 indistinguishable 4-person committees. Each committee will have one chairperson and one secretary. The number of different ways to make these assignments can be written as $3^r M$, where r and M are positive integers and M is not divisible by 3. What is r ?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (A): The 16 people can be partitioned into the 4 committees, each of size 4, in

$$\frac{16!}{(4!)^5}$$

ways; four of the 4! factors come from permuting the members of the committees and one 4! factor comes from permuting the committees. Then there are 4^4 ways to choose the four chairpersons and 3^4 ways to choose the four secretaries. This gives a total of

$$\frac{16! \cdot 4^4 \cdot 3^4}{(4!)^5}$$

assignments. The numerator has 1 factor of 3 in each of 15, 12, 6, and 3; it has 2 factors of 3 in 9 and 4 factors in 3^4, a total of 10. The denominator has 5 factors of 3. Thus $r = 10 - 5 = 5$.

OR

There are $16 \cdot 15$ ways to choose the chairperson and secretary of the first committee, $14 \cdot 13$ ways to choose the chairperson and secretary of the second committee, $12 \cdot 11$ ways to choose the chairperson and secretary of the third committee, and $10 \cdot 9$ ways to choose the chairperson and secretary of the fourth committee. There are then $\frac{8\cdot 7}{2} = 28$ ways to choose the remaining members of the first committee, $\frac{6\cdot 5}{2} = 15$ ways to choose the remaining members of the second committee, $\frac{4\cdot 3}{2} = 6$ ways to choose the remaining members of the third committee, and $\frac{2\cdot 1}{2} = 1$ way to choose the remaining members of the fourth committee. There are $4!$ ways to account for the fact that the committees are indistinguishable. This gives a total of

$$\frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 28 \cdot 15 \cdot 6 \cdot 1}{4!}$$

assignments. There are $1 + 1 + 2 + 1 + 1 = 6$ factors of 3 in the numerator and 1 factor of 3 in the denominator, so $r = 6 - 1 = 5$.

Note: The number of ways to make the assignments is $54{,}486{,}432{,}000 = 3^5 \cdot 224{,}224{,}000$.

23. The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. What is

$$\frac{F_2}{F_1} + \frac{F_4}{F_2} + \frac{F_6}{F_3} + \cdots + \frac{F_{20}}{F_{10}}\ ?$$

(A) 318 **(B)** 319 **(C)** 320 **(D)** 321 **(E)** 322

Answer (B): The Fibonacci sequence starts out

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, \ldots,$$

so the given sum is

$$\frac{1}{1} + \frac{3}{1} + \frac{8}{2} + \frac{21}{3} + \frac{55}{5} + \frac{144}{8} + \frac{377}{13} + \frac{987}{21} + \frac{2584}{34} + \frac{6765}{55},$$

which equals $1 + 3 + 4 + 7 + 11 + 18 + 29 + 47 + 76 + 123 = 319$.

OR

The Fibonacci sequence starts out $1, 1, 2, 3, 5, 8, 13, 21, 34, 55$, so the given sum starts out

$$\frac{1}{1} + \frac{3}{1} + \frac{8}{2} + \frac{21}{3} + \frac{55}{5} = 1 + 3 + 4 + 7 + 11.$$

It appears that these summands satisfy the same recurrence relation, namely

$$\frac{F_{2n}}{F_n} = \frac{F_{2(n-1)}}{F_{n-1}} + \frac{F_{2(n-2)}}{F_{n-2}}.$$

With the initial conditions $1, 3$ instead of $1, 1$, the sequence

$$(L_n) = \left(\frac{F_{2n}}{F_n}\right)$$

is known as the Lucas sequence. If the recurrence above is correct, then the required sum is

$$1 + 3 + 4 + 7 + 11 + 18 + 29 + 47 + 76 + 123 = 319.$$

To prove the identity for (L_n) displayed above, recall Binet's formula, $F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n)$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ are the roots of the polynomial $x^2 - x - 1$. Then

$$L_n = \frac{F_{2n}}{F_n} = \frac{\phi^{2n} - \psi^{2n}}{\phi^n - \psi^n} = \phi^n + \psi^n.$$

Therefore

$$\begin{aligned} L_{n-1} + L_{n-2} &= \phi^{n-1} + \phi^{n-2} + \psi^{n-1} + \psi^{n-2} \\ &= \phi^{n-2}(\phi + 1) + \psi^{n-2}(\psi + 1) \\ &= \phi^{n-2} \cdot \phi^2 + \psi^{n-2} \cdot \psi^2 \\ &= \phi^n + \psi^n = L_n. \end{aligned}$$

24. Let

$$P(m) = \frac{m}{2} + \frac{m^2}{4} + \frac{m^4}{8} + \frac{m^8}{8}.$$

How many of the values $P(2022)$, $P(2023)$, $P(2024)$, and $P(2025)$ are integers?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (E): Let $Q(m) = 8P(m) = 4m + 2m^2 + m^4 + m^8$. Because the coefficients of Q are integers, it follows that if $a \equiv b \pmod 8$, then $Q(a) \equiv Q(b) \pmod 8$. It suffices to show that the 8 numbers $Q(-3), Q(-2), Q(-1), \ldots, Q(4)$ are all divisible by 8. If m is even, then each of the monomials of $Q(m)$ is divisible by 8. If $m = \pm 1$, then $Q(m) = \pm 4 + 4 \equiv 0 \pmod 8$. If $m = \pm 3$, then $m^2 = 9 \equiv 1 \pmod 8$, which implies that $m^4 \equiv 1 \pmod 8$, and so also that $m^8 \equiv 1 \pmod 8$. Hence $Q(\pm 3) \equiv \pm 12 + 2 + 1 + 1 \equiv 0 \pmod 8$.

Therefore $8P(m)$ is divisible by 8 for all integers m, which implies that $P(m)$ is an integer for all m. In particular, all 4 of the given values of $P(m)$ are integers.

OR

Let $Q(m)$ be defined as in the first solution, and note that $Q(m)$ is divisible by 8 if m is even. To treat odd m, write

$$\begin{aligned} Q(m) &= 8m + 4(m^2 - m) + 2m^2(m^2 - 1) + m^4(m^4 - 1) \\ &= 8m + 4m(m-1) + 2m^2(m+1)(m-1) + m^4(m^2+1)(m+1)(m-1) \end{aligned}$$

and note that because $m+1$, $m-1$, and m^2+1 are all even, each term has at least three factors of 2. The solution concludes as above.

OR

Another way to see that $Q(m)$ is divisible by 8 when m is odd in the second solution is to apply more general facts from number theory. Fermat's Little Theorem asserts that if p is prime, then $a^p \equiv a \pmod p$ for all integers a. In particular, $m^2 \equiv m \pmod 2$. Euler's Totient Theorem asserts that if

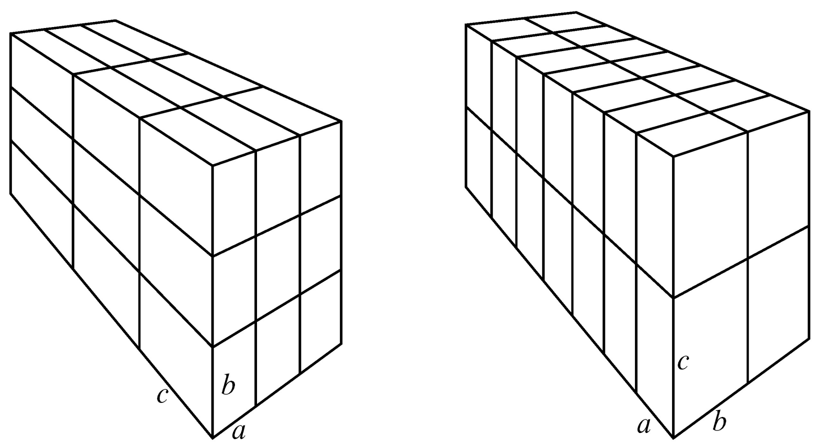
$\gcd(a, q) = 1$, then $a^{\phi(q)} \equiv 1 \pmod{q}$, where $\phi(q)$ is the number of positive integers less than q that are relatively prime to q. Because $\phi(4) = 2$, it follows that $m^2 \equiv 1 \pmod{4}$ when m is odd. Also, $\phi(8) = 4$, so $m^4 \equiv 1 \pmod{8}$ if m is odd.

Note: Suppose that a necklace is to be made by placing n beads, equally spaced, on a circular ring. Each bead can be any one of m different colors. Two necklaces are regarded as being the same if they differ only by rotation. In 1892 Captain Percy Alexander MacMahon showed that the number of such necklaces is

$$\frac{1}{n}\sum_{d|n}\phi\left(\frac{n}{d}\right)m^d.$$

He acknowledged its prior discovery by another soldier, Monsieur le Colonel Charles Paul Narcisse Moreau. Thus $P(m)$ counts the number of necklaces with $n = 8$ beads and therefore must be an integer. This is discussed in *Concrete Mathematics* by Graham, Knuth, and Patashnik.

25. Each of 27 bricks (right rectangular prisms) has dimensions $a \times b \times c$, where a, b, and c are pairwise relatively prime positive integers. These bricks are arranged to form a $3 \times 3 \times 3$ block, as shown on the left below. A 28th brick with the same dimensions is introduced, and these bricks are reconfigured into a $2 \times 2 \times 7$ block, shown on the right. The new block is 1 unit taller, 1 unit wider, and 1 unit deeper than the old one. What is $a + b + c$?



(A) 88 **(B)** 89 **(C)** 90 **(D)** 91 **(E)** 92

Answer (E): Without loss of generality, assume $a < b < c$. Comparing the figures and considering the change in orientation gives rise to the equations $3a + 1 = 2b$, $3b + 1 = 2c$, and $3c + 1 = 7a$. To solve this system of linear equations, use the first two equations to write a and c in terms of b, namely $a = \frac{2}{3}b - \frac{1}{3}$ and $c = \frac{3}{2}b + \frac{1}{2}$. Substituting these into the third equation gives $\frac{9}{2}b + \frac{3}{2} + 1 = \frac{14}{3}b - \frac{7}{3}$. Multiplying both sides by 6 yields $27b + 9 + 6 = 28b - 14$, which shows that $b = 29$. Back substituting then gives $a = 19$ and $c = 44$. The requested sum is $19 + 29 + 44 = 92$.

Note: This problem was inspired by a lovely puzzle created by Thomas O'Beirn called "Melting Block".

Problems and solutions were contributed by Theodore Alper, David Altizio, Risto Atanasov, Bela Bajnok, Chris Bolognese, Silva Chang, Steven Davis, Thinula De Silva, Steve Dunbar, Marta Eso, Zuming Feng, Zachary Franco, Peter Gilchrist, Gary Gordon, Jon Graetz, Andrea Grof, Jerrold Grossman, Thomas Howell, Chris Jeuell, Daniel Jordan, Jonathan Kane, Ed Keppelman, Azar Khosravani, Sergey Levin, Joseph Li, Ken Liu, Ioana Mihaila, Hugh Montgomery, Bryan Nevarez, Lucian Sega, Kate Thompson, Agnes Tuska, David Wells, Kathleen Wong, Andrew Wu, Dai Yang, Carl Yerger, Jesse Zhang, and Alexander Zhu.