2017 AMC 10A

1 What is the value of $(2(2(2(2(2+1)+1)+1)+1)+1)+1)$? (2017 AMC 10A Problem, Question#1)

A. 70 B. 97 C. 127 D. 159 E. 729

Answer C

Solution Notice this is the term a_6 in a recursive sequence, defined recursively

$a_2 = 3*2+1 = 7$.

$a_3 = 7*2+1 = 15$.

$a_4 = 15*2+1 = 31$.

$a_5 = 31*2+1 = 63$.

as$a_1 = 3$, $a_n = 2a_{n-1}+1$.Thus: $a_6 = 63*2+1 = (C)127$.

Starting to compute the inner expressions, we see the results are 1, 3, 7, 15,.... This is always 1 less than a power of 2. The only admissible answer choice by this ruleis thus $(C)127$.

Working our way from the innermost parenthesis outwards and directly computing, we have $(C)127$.

If you distribute this you get a sum of the powers of 2. The largest power of 2 in the series is 64, so the sumis $(C)127$.

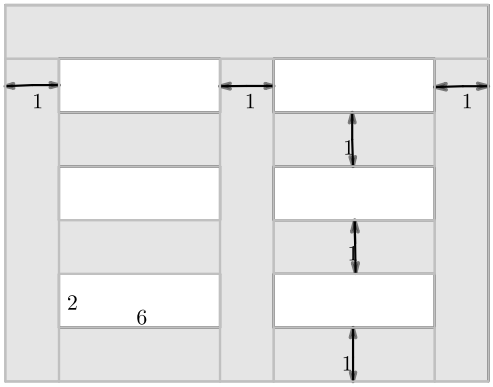
2 Pablo buys popsicles for his friends. The store sells single popsicles for $\$1$ each, 3–popsicle boxes for $\$2$ each, and 5–popsicle boxes for $\$3$. What is the greatest number of popsicles that Pablo can buy with $\$8$? (2017 AMC 10A Problem, Question#2)

A. 8 B. 11 C. 12 D. 13 E. 15

Answer D

Solution $\$3$ boxes give us the most popsicles/dollar, so we want to buy as many of those as possible. After buying 2, we have $\$2$ left. We cannot buy a third $\$3$ box, so we opt for the $\$2$ box instead (since it has a higher popsicles/dollar ratio than the $\$1$ pack). We're now out of money. We bought $5+5+3=13$ popsicles, so the answer is $(\mathbf{D})13$.

3 Tamara has three rows of two 6–feet by 2–feet flower beds in her garden. The beds are separated and also surrounded by 1–foot–wide walkways, as shown on the diagram. What is the total area of the walkways, in square feet? (2017 AMC 10A Problem, Question#3)



A. 72 B. 78 C. 90 D. 120 E. 150

Answer B

Solution Finding the area of the shaded walkway can be achieved by computing the total area of Tamara's garden and then subtracting the combined area of her six flower beds.

Since the width of Tamara's garden contains three margins, the total width is $2\cdot 6+3\cdot 1=15$ feet.

Similarly, the height of Tamara's garden is $3\cdot 2+4\cdot 1=10$ feet.

Therefore, the total area of the garden is $15\cdot 10=150$ square feet.

Finally, since the six flower beds each have an area of $2\cdot 6=12$ square feet, the area we seek is $150-6\cdot 12$, and our answer is $(\mathbf{B})78$.

4 Mia is "helping" her mom pick up 30 toys that are strewn on the floor. Mia's mom manages to put 3 toys into the toy box every 30 seconds, but each time immediately after those 30 seconds have elapsed, Mia takes 2 toys out of the box. How much time, in minutes, will it take Mia and her mom to put all 30 toys into the box for the first time? (2017 AMC 10A Problem, Question#4)

A. 13.5 B. 14 C. 14.5 D. 15 E. 15.5

Answer B

Solution Every 30 seconds $3-2=1$ toys are put in the box, so after $27 \cdot 30$ seconds there will be 27 toys in the box. Mia's mom will then put 3 toys into to the box and we have our total amount of time to be $27 \cdot 30+30=840$ seconds, which equals 14 minutes. $\mathbf{(B)14}$

Though Mia's mom places 3 toys every 30 seconds, Mia takes out 2 toys right after. Therefore, after 30 seconds, the two have collectively placed 1 toy into the box. Therefore by 13.5 minutes, the two would have placed 27 toys into the box. Therefore, at 14 minutes, the two would have placed 30 toys into the box. Though Mia may take 2 toys out right after, the number of toys in the box first reaches 30 by 14 minutes. $\mathbf{(B)14}$

5 The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers? (2017 AMC 10A Problem, Question#5)

A. 1 B. 2 C. 4 D. 8 E. 12

Answer C

Solution Let the two real numbers be x, y. We are given that $x+y=4xy$, and dividing both sides by xy, $\frac{x}{xy}+\frac{y}{xy}=4$.

$\frac{1}{y}+\frac{1}{x}=\mathbf{(C)4}$,

Note: we can easily verify that this is the correct answer; for example, $\frac{1}{2}$ and $\frac{1}{2}$ work, and the sum of their reciprocals is 4.

Instead of using algebra, another approach at this problem would be to notice the fact that one of the nonzero numbers has to be a fraction. See for yourself. And by looking into fractions, we immediately see that $\frac{1}{3}$ and 1 would fit the rule.(C)4

Notice that from the information given above, $x+y=4xy$.

Because the sum of the reciprocals of two numbers is just the sum of the two numbers over the product of the two numbers or $\frac{x+y}{xy}$.

We can solve this by substituting $x+y \Rightarrow 4xy$.

Our answer is simplya $\frac{4xy}{xy} \Rightarrow 4$.

Theretve, he answeris $(C)4$.

Let the two numbers be a and b, respectively. We wish to find $\frac{1}{a}+\frac{1}{b}$. Note that $\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}$. We are given that $a+b=4ab$.

Subsituting, we have $\frac{a+b}{ab}=\frac{4ab}{ab}=(C)4$

6 Ms. Carroll promised that anyone who got all the multiple choice questions right on the upcoming exam would receive an A on the exam. Which of these statements necessarily follows logically? (2017 AMC 10A Problem, Question#6)

A. If Lewis did not receive an A, then he got all of the multiple choice questions wrong.

B. If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.

C. If Lewis got at least one of the multiple choice questions wrong, then he did not receive an A.

D. If Lewis received an A, then he got all of the multiple choice questions right.

E. If Lewis received an A, then he got at least one of the multiple choice questions right.

Answer B

Solution

Rewriting the given statement: "if someone got all the multiple choice questions right on the upcoming exam then he or she would receive an A on the exam." If that someone is Lewis the statement becomes: "if Lewis got all the multiple choice questions right, then he got an A on the exam." The contrapositive: "If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong (did not get all of them right)" must also be true leaving B as the correct answer. B is also equivalent to the contrapositive of the original statement, which implies that it must be true, so the answer is

(B) If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.

Note that Answer choice (B) is the contrapositive of the given statement. Weknow that

contrapositive is always true if the given statement is true.

7 Jerry and Silvia wanted to go from the southwest corner of a square field to the northeast corner. Jerry walked due east and then due north to reach the goal, but Silvia headed northeast and reached the goal walking in a straight line. Which of the following is closest to how much shorter Silvia's trip was, compared to Jerry's trip? (2017 AMC 10A Problem, Question#7)

A. 30% B. 40% C. 50% D. 60% E. 70%

Answer A

Solution Let j represent how far Jerry walked, and s represent how far Sylvia walked. Since the field is a square, and Jerry walked two sides of it, while Silvia walked the diagonal, we can simply define the side of the square field to be one, and find the distances they walked. Since Jerry walked two sides, $j = 2$ Since Silvia walked the diagonal, she walked the hypotenuse of a 45, 45, 90 triangle with leg length 1. Thus, $s = \sqrt{2} = 1.414\ldots$. We can then $\frac{j-s}{j} \approx \frac{2-1.4}{2} = 0.3 \Rightarrow (\mathrm{A})30\%$

8 At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur? (2017 AMC 10A Problem, Question#8)

A. 240 B. 245 C. 290 D. 480 E. 490

Answer B

Solution Each one of the ten people has to shake hands with all the 20 other people they don't know. So $10 \cdot 20 = 200$. From there, we calculate how many handshakes occurred between the people who don't know each other. This is simply counting how many ways to choose two people to shake hands, or $\binom{10}{2} = 45$. Thus the answer is $200 + 45 = (\mathbf{B})245$.

We can also use complementary counting. First of all, $\binom{30}{2} = 435$ handshakes or hugs occur. Then, if we can find the number of hugs, then we can subtract it from 435 to find the handshakes. Hugs only happen between the 20 people who know each other, so there are $\binom{20}{2} = 190$ hugs. $435 - 190 = (\mathbf{B})245$.

We can focus on how many handshakes the 10 people get.

The 1st person gets 29 handshakes.

2nd gets 28.

.....

And the 10th receives 20 handshakes.

We can write this as the sum of an arithmetic sequence.

$\frac{10(20+29)}{2} \Rightarrow 5(49) \Rightarrow 245$. Therefore, the answer is $(\mathbf{B})245$.

9 Minnie rides on a flat road at 20 kilometers per hour (kph), downhill at 30kph, and uphill at 5kph. Penny rides on a flat road at 30kph, downhill at 40kph, and uphill at 10kph. Minnie goes from town A to town B, a distance of 10km all uphill, then from town B to town C, a distance of 15km all downhill, and then back to town A, a distance of 20km on

the flat. Penny goes the other way around using the same route. How many more minutes does it take Minnie to complete the $45-\text{km}$ ride than it takes Penny? (2017 AMC 10A Problem, Question#9)

A. 45 B. 60 C. 65 D. 90 E. 95

Answer C

Solution The distance from town A to town B is 10km uphill, and since Minnie rides uphill at a speed of 5kph, it will take her 2 hours. Next, she will ride from town B to town C, a distance of 15km all downhill. Since Minnie rides downhill at a speed of 30kph, it will take her half an hour. Finally, she rides from town C back to town A, a flat distance of 20km. Minnie rides on a flat road at 20kph, so this will take her 1 hour. Her entire trip takes her 3.5 hours. Secondly, Penny will go from town A to town C, a flat distance of $20km$. Since Penny rides on a flat road at 30kph, it will take her $\frac{2}{3}$ of an hour. Next Penny will go from town C to town B, which is uphill for Penny. Since Penny rides at a speed of 10kph uphill, and town C and B are 15km apart, it will take her 1.5 hours. Finally, Penny goes from Town B back to town A, a distance of 10km downhill. Since Penny rides downhill at 40kph, it will only take her $\frac{1}{4}$ of an hour. In total, it takes her $\frac{29}{12}$ hours, which simplifies to 2 hours and 25 minutes. Finally, Penny's 2 Hour 25 Minute trip was $(\text{C})65$ minutes less than Minnie's 3 Hour 30 Minute Trip.

10 Joy has 30 thin rods, one each of every integer length from 1cm through 30cm. She places the rods with lengths 3cm, 7cm, and 15cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod? (2017 AMC 10A Problem, Question#10)

A. 16 B. 17 C. 18 D. 19 E. 20

Answer B

Solution The triangle inequality generalizes to all polygons, so $x < 3 + 7 + 15$ and $x + 3 + 7 > 15$ to get $5 < x < 25$. Now, we know that there are 19 numbers between 5 and 25 exclusive, but we must subtract 2 to account for the 2 lengths already used that are between those numbers, which gives $19 - 2 = (\mathrm{B})17$.

11 The region consisting of all points in three−dimensional space within 3 units of line segment $\overline{AB}$ has volume 216π . What is the length AB? (2017 AMC 10A Problem, Question#11)



A. 6 B. 12 C. 18 D. 20 E. 24

Answer D

Solution In order to solve this problem, we must first visualize what the region contained looks like. We know that, in a three dimensional plane, the region consisting of all points within 3 units of a point would be a sphere with radius 3. However, we need to find the region containing all points within 3 units of a segment. It can be seen that our region is a cylinder with two hemispheres on either end. We know the volume of our region, so we set up the following equation (the volume of our cylinder + the volume of our two hemispheres will equal 216π): $\frac{4\pi}{3} \cdot 3^3 + 9\pi x = 216\pi$, where x is equal to the length of our line segment. Solving, we find that $x = (\mathrm{D})20$. Because this is just a cylinder and 2 "half spheres", and the radius is 3, the volume of the 2 half spheres is $4(3^3)/3\pi = 36\pi$. Since we also know that the volume of this whole thing is 216π , we do $216 - 36$ to get 180π as the area of the cylinder. Thus the height is 180π over the base, or $180\pi/9\pi = 20$, D.

12 Let S be a set of points (x, y) in the coordinate plane such that two of the three quantities 3, $x+2$, and $y-4$ are equal and the third of the three quantities is no greater than this common value. Which of the following is a correct description for S? (2017 AMC 10A Problem, Question#12)

A. a single point

B. two intersecting lines

C. three lines whose pairwise intersections are three distinct points

D. a triangle

E. three rays with a common endpoint

Answer E

Solution If the two equal values are 3 and $x+2$, then $x=1$. Also, $y-4 \leq 3$ because 3 is the common value. Solving for y, we get $y \leq 7$.Therefore the portion of the line $x=1$ where $y \leq 7$ is part of S. This is a ray with an endpoint of $(1, 7)$.

Similar to the process above, we assume that the two equal values are 3 and $y-4$. Solving the equation $3=y-4$ then $y=7$. Also, $x+2 \leq 3$ because 3 is the common value. Solving for x, we get $x \leq 1$. Therefore the portion of the line $y=7$ where $x \leq 1$ is also part of S. This is another ray with the same endpoint as the above ray: $(1,7)$.

If x+2 and$y-4$ are the two equal values, then $x+2=y-4$. Solving the equation for y, we get $y=x+6$. Also $3 \leq y-4$ because $y-4$ is one way to express the common value. Solving for y, we get $y \geq 7$. Therefore the portion of the line $y=x+6$ where $y \geq 7$ is part of S like the other two rays. The lowest possible value that can be achieved is also $(1, 7)$.

Since S is made up of three rays with common endpoint $(1,7)$, the answer is

$(\mathbf{E})$ three rays with a common endpoint

13

Define a sequence recursively by $F_0 = 0$, $F_1 = 1$, and $F_n =$ the remainder when $F_{n-1} + F_{n-2}$ is divided by 3, for all $n \geq 2$. Thus the sequence starts $0, 1, 1, 2, 0, 2, \cdots$. What is $F_{2017} + F_{2018} + F_{2019} + F_{2020} + F_{2021} + F_{2022} + F_{2023} + F_{2024}$? (2017 AMC 10A Problem, Question#13)

A. 6 B. 7 C. 8 D. 9 E. 10

Answer D

Solution A patten starts to emerge as the function is continued. The repeating patten is $0, 1, 1, 2, 0, 2, 2, 1 \ldots$. The problem asks for the sum of eight consecutive terms in the sequence.

Because there are eight numbers in the repeating sequence, we just need to find the sum of the numbers in the sequence, which is $(\mathbf{D})9$.

14 Every week Roger pays for a movie ticket and a soda out of his allowance. Last week, Roger's allowance was A dollars. The cost of his movie ticket was 20% of the difference between A and the cost of his soda, while the cost of his soda was 5% of the difference between A and the cost of his movie ticket. To the nearest whole percent, what fraction of A did Roger pay for his movie ticket and soda? (2017 AMC 10A Problem, Question#14)

A. 9% B. 19% C. 22% D. 23% E. 25%

Answer D

Solution Let $m =$ cost of movie ticket,

Let $s =$cost of soda,

We can create two equations:

$m = \frac{1}{5}(A - s)$.

$s = \frac{1}{20}(A - m)$.

Substituting we get:

$m = \frac{1}{5}(A - \frac{1}{20}(A - m))$.

which yields:

$m = \frac{19}{99}$A.

Now we can find s and we get:

$s = \frac{4}{99}A$.

Since we want to find what fraction of A did Roger pay for his movie ticket and soda, we

add m and s to get:

$\frac{19}{99}A + \frac{4}{99}A \Rightarrow (\mathrm{D})23$ %.

15 Chloé chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chloé's number? (Assume they cannot be equal). (2017 AMC 10A Problem, Question#15)

A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{3}{4}$ D. $\frac{5}{6}$ E. $\frac{7}{8}$

Answer C

Solution Denote "winning" to mean "picking a greater number". There is a $\frac{1}{2}$ chance that Laurent chooses a number in the interval $[2018, 4034]$ in this case, Chloé cannot possibly win, since the maximum number she can pick is 2017. Otherwise, if Laurent picks a number in the interval $[0, 2017]$, with probability $\frac{1}{2}$ then the two people are symmetric, and each has a $\frac{1}{2}$ chance of winning. Then the total probabilty is $\frac{1}{2} * 1 + \frac{1}{2} * \frac{1}{2} = (\mathrm{C})\frac{3}{4}$.

We can use geometric probability to solve this. Suppose a point (x, y) lies in the $xy-$plane. Let x be Chloe's number and y be Laurent's number. Then obviously we want $y > x$, which basically gives us a region above a line. We know that Chloe's number is in the interval $[0, 2017]$ and Laurent's number is in the interval $[0, 4034]$, so we can

create a rectangle in the plane, whose length is 2017 and whose width is 4034. Drawing it out, we see that it is easier to find the probability that Chloe's number is greater than Laurent's number and subtract this probability from 1. The probability that Chloe's number is larger than Laurent's number is simply the area of the region under the line $y > x$, which is $\frac{2017 \cdot 2017}{2}$. Instead of bashing this out we know that the rectangle has area $2017 \cdot 4034$. So the probability that Laurent has a smaller number is $\frac{2017 \cdot 2017}{2 \cdot 2017 \cdot 4034}$. Simplifying the expression yields $\frac{1}{4}$ and so $1 - \frac{1}{4} = (\mathrm{C})\frac{3}{4}$. Scale down by 2017 to get that Chloe picks from $[0, 1]$ and Laurent picks from $[0, 2]$. There are an infinite number of cases for the number that Chloe picks, but they are all centered around the average of 0.5. Therefore, Laurent has a range of 0.5 to 2 to pick from, on average, which is a length of $2 - 0.5 = 1.5$ out of a total length of $2 - 0 = 2$. Therefore, the probability is $1.5/2 = 15/20 = 3/4(\mathrm{C})$.

16 There are 10 horses, named Horse 1, Horse 2, ..., Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds. The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of T? (2017 AMC 10A Problem, Question#16)

A. 2 B. 3 C. 4 D. 5 E. 6

Answer B

Solution If we have horses, a_1, a_2, ..., a_n, then any number that is a multiple of the all those numbers is a time when all horses will meet at the starting point. The least of these numbers is the LCM. To minimize the LCM, we need the smallest primes, and we

need to repeat them a lot. By inspection, we find that $\text{LCM}(1,2,3,2\cdot 2,2\cdot 3)=12$. Finally, $1+2=(\text{B})3$.

We are trying to find the smallest number that has 5 one−digit divisors. Therefore we try to find the LCM for smaller digits, such as 1, 2, 3, or 4. We quickly consider 12 since it is the smallest number that is the LCM of 1, 2, 3 and 4. Since 12 has 5 single−digit divisors, namely 1, 2, 3, 4, and 6, our answer is $1+2=(\text{B})3$.

17 Distinct points P, Q, R, S lie on the circle $x^2+y^2=25$ and have integer coordinates. The distances PQ and RS are irrational numbers. What is the greatest possible value of the ratio $\frac{PQ}{RS}$? (2017 AMC 10A Problem, Question#17)

A. 3 B. 5 C. $3\sqrt{5}$ D. 7

E. $5\sqrt{2}$

Answer D

Solution Because P, Q, R, and S are lattice points, there are only a few coordinates that actually satisfy the equation. The coordinates are $(\pm 3,\pm 4)$, $(\pm 4,\pm 3)$, $(0,\pm 5)$, and$(\pm 5,0)$. We want to maximize PQ and minimize RS. They also have to be the square root of something, because they are both irrational. The greatest value of PQ happens when P and Q are almost directly across from each other and are in different quadrants. For example, the endpoints of the segment could be $(-4,3)$ and $(3,-4)$ because the two points are almost across from each other. Another possible pair could be $(-4,3)$ and $(5,0)$. To find out which segment is longer, we have to compare the distances from their endpoints to a diameter (which must be the longest possible segment). The closest diameter would be from $(-4,3)$ to $(4,-3)$. The distance between $(3,-4)$ and $(4,-3)$ is shorter than the distance between $(5,0)$ and $(4,-3)$ Therefore, the segment from $(-4,3)$ to $(3,-4)$ is the longest attainable. The least value of RS is when the two endpoints are in the same quadrant and are very close to each other. This can occur when, for example, R is $(3,4)$ and S is $(4,3)$. They are in the same

quadrant and no other point on the circle with integer coordinates is closer to the point $(3,4)$ than $(4,3)$. Using the distance formula, we get that PQ is $\sqrt{98}$ and that RS is $\sqrt{2}$. $\frac{\sqrt{98}}{\sqrt{2}} = \sqrt{49} = (\mathrm{D})7$.

18 Amelia has a coin that lands on heads with probability $\frac{1}{3}$, and Blaine has a coin that lands on heads with probability $\frac{2}{5}$. Amelia and Blaine alternately toss their coins until someone gets a head; the first one to get a head wins. All coin tosses are independent. Amelia goes first. The probability that Amelia wins is $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $q-p$? (2017 AMC 10A Problem, Question#18)

A. 1 B. 2 C. 3 D. 4 E. 5

Answer D

Solution Let P be the probability Amelia wins. Note that $P =$ chance she wins on her first turn $+$ chance she gets to her turn again$\cdot P$, as if she gets to her turn again, she is back where she started with probability of winning P. The chance she wins on her first turn is $\frac{1}{3}$. The chance she makes it to her turn again is a combination of her failing to win the first turn$-\frac{2}{3}$ and Blaine failing to win $-\frac{3}{5}$. Multiplying gives us $\frac{2}{5}$. Thus, $P = \frac{1}{3} + \frac{2}{5}P$. Therefore, $p = \frac{9}{5}$, so the answer is $9-5 = (\mathrm{D})4$.

Let P be the probability Amelia wins. Note that $P =$ chance she wins on her frst turn $+$ chance she gets to her second turn $\cdot\frac{1}{3}+$ chance she gets to her third turn $\cdot\frac{1}{3}\ldots$. This can be represented by an infinite geometric series.

$p = \frac{\frac{1}{3}}{1-\frac{2}{3}\cdot\frac{3}{6}} = \frac{\frac{1}{3}}{1-\frac{2}{5}} = \frac{\frac{1}{3}}{\frac{3}{5}} = \frac{1}{3}\cdot\frac{5}{3} = \frac{5}{9}$. Therefre $P = \frac{5}{9}$, so the answer is $9-5 = (\mathrm{D})4$.

19 Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these

conditions? (2017 AMC 10A Problem, Question#19)

A. 12 B. 16 C. 28 D. 32 E. 40

Answer C

Solution For notation purposes, let Alice be A, Bob be B, Carla be C, Derek be D, and Eric be E. We can split this problem up into two cases:

Case 1: A sits on an edge seat.

Then, since B and C can't sit next to A, that must mean either D or E sits next to A. After we pick either D or E, then either B or C must sit next to D/E. Then, we can arrange the two remaining people in two ways. Since there are two different edge seats that A can sit in, there are a total of $2 \cdot 2 \cdot 2 \cdot 2 = 16$.

Case 2: A does not sit in an edge seat.

In this case, then only two people that can sit next to A are D and E, and there are two ways to permute them, and this also handles the restriction that D can't sit next to E. Then, there are two ways to arrange B and C, the remaining people. However, there are three initial seats that A can sit in, so there are $3 \cdot 2 \cdot 2 = 12$ seatings in this case. Adding up all the cases, we have $16 + 12 = (C)28$.

Label the seats 1 through 5. The number of ways to seat Derek and Eric in the five seats with no restrictions is $5 * 4 = 20$. The number of ways to seat Derek and Eric such that they sit next to each other is 8 (which can be figure out quickly), so the number of ways such that Derek and Eric don't sit next to each other is $20 - 8 = 12$.

Note that once Derek and Eric are seated, there are three cases.

The first case is that they sit at each end. There are two ways to seat Derek and Eric. But this is impossible because then Alice, Bob, and Carla would have to sit in some order in the middle three seats which would lead to Alice sitting next to Bob or Carla, a contradiction. So this case gives us 0 ways.

Another possible case is if Derek and Eric seat in seats 2 and 4 in some order. There are 2 possible ways to seat Derek and Eric like this. This leaves Alice, Bob, and Carla to sit in any order in the remaining three seats. Since no two of these three seats are

consecutive, there are $3! = 6$ ways to do this. So the second case gives us $2 * 6 = 12$ total ways for the second case.

The last case is if once Derek and Eric are seated, exactly one pair of consecutive seats is available. There are $12 - 2 - 2 = 8$ ways to seat Derek and Eric like this. Once they are seated like this, Alice cannot not sit in one of the two consecutive available seats without sitting next to Bob and Carla. So Alice has to sit in the other remaining chair. Then, there are two ways to seat Bob and Carla in the remaining two seats (which are consecutive). So this case gives us $8 * 2 = 16$ ways.

So in total there are $12 + 16 = 28$. So ouranswer is $(\mathrm{C})28$.

20 Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. Which of the following could be the value of $S(n+1)$? (2017 AMC 10A Problem, Question#20)

A. 1 B. 3 C. 12 D. 1239 E. 1265

Answer D

Solution Note that $n \equiv S(n) (\mathrm{mod} 9)$. This can be seen from the fact that $\sum_{k=0}^{n} 10^k a_k \equiv \sum_{k=0}^{n} a_k \pmod 9$. Thus, if $S(n) = 1274$, then $n = 5 (\mathrm{mod} 9)$, and thus $n + 1 = S(n+1) \equiv 6 (\mathrm{mod} 9)$. The only answer choice that is $6 (\mathrm{mod} 9)$ is $(\mathrm{D})1239$.

One divisibility rule for division that we can use in the problem is that a multiple of 9 has its digit always add up to a multiple of 9. We can find out that the least number of digits the number N is 142, with $1419's$ and 15, assuming the rule above. No matter what arrangement or different digits we use, the divisor rule stays the same. To make the problem simpler, we can just use the $1419's$ and 15. By randomly mixing the digits up, we are likely to get: $9999 \ldots 9995999 \ldots 9999$. By adding 1 to this number, we get: $9999 \ldots . 9996000 \ldots . 0000$. Knowing that this number is ONLY divisible by 9 when 6 is subtracted,

we can subtract 6 from every available choice, and see if the number is divisible by 9 afterwards. After subtracting 6 from every number, we can conclude that 1233 (originally 1239) is the only number divisible by 9. So our answer is $(\mathrm{D})1239$.

The number n can be viewed as having some unique digits in the front, following by a certain number of nines. We can then evaluate each potential answer choice.

If$(\mathrm{A})1$ is correct, then n must be some number $99999999\ldots9$, because when we add one to $99999999\ldots9$ we get $10000000\ldots00$. Thus, if 1 is the correct answer, then the equation $9x=1274$ must have an integer solution (i.e. 1274 must be divisible by 9). But since it does not, 1 is not the correct answer.

If $(\mathrm{B})3$ is corret, then n must be some number $29999999\ldots9$, because when we add one to $29999999\ldots9$, we get $30000000\ldots00$.Thus, if 3 is the correct answer, then the equation $2+9x=1274$ must have an integer solution. But since it does not, 3 is not the correct answer.

Based on what we have done for evaluating the previous two answer choices, we can create an equation we can use to evaluate the final three possibilities.Notice that if $S(n+1)=N$, then n must be a number whose initial digits sum to $N-1$, and whose other, terminating digits, are all 9. Thus, we can evaluate the three final possibilities by seeing if the equation $(N-1)+9x=1274$ has an integer solution.

The equation does not have an integer soutionfor $N=12$, so $(\mathrm{C})12$ is not correct.

However, the equation does have an integer solutionfor $N=1239(x=4)$ so $(\mathrm{D})1239$ is the answer.

21 A square with side length x is inscribed in a right triangle with sides of length 3, 4, and 5 so that one vertex of the square coincides with the right−angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length 3, 4, and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

(2017 AMC 10A Problem, Question#21)

A. $\frac{12}{13}$

B. $\frac{35}{37}$

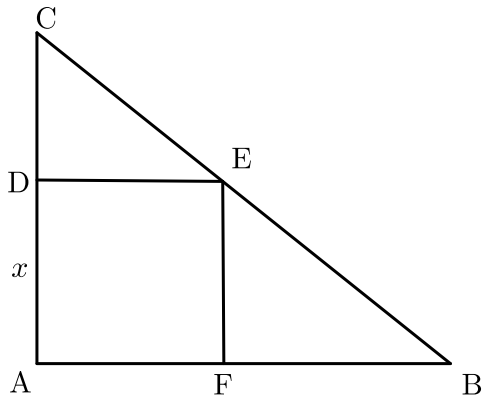
C. 1

D. $\frac{37}{35}$

E. $\frac{13}{12}$

Answer D

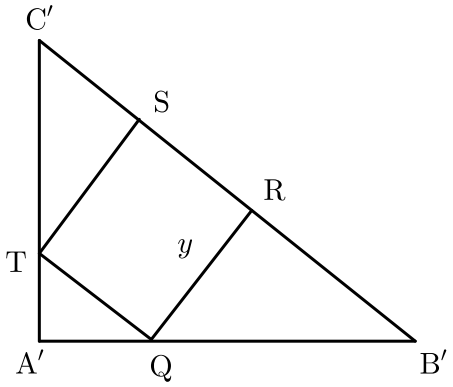
Solution Analyze the first right triangle.



Note that $\triangle ADC$ ant $\triangle FBE$ are similar, so $\frac{BF}{FE}=\frac{AB}{AC}$.This can be written as $\frac{4-x}{x}=\frac{4}{3}$.

Soving, $x=\frac{12}{7}$

Now we analyze the second triangle.



Similarly, $\triangle A'D'C'$ and $\triangle RB'Q$are similar, so $RB'=\frac{3}{4}y$ and $C'S=\frac{3}{4}y$.

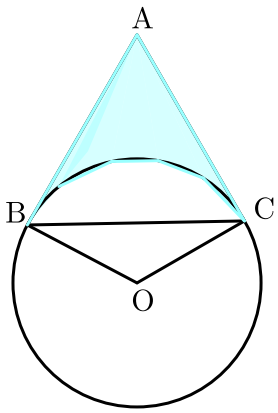
Thus, $C'B'=C'S+SR+RB'=\frac{4}{3}y+y+\frac{3}{4}y=5$. Solving for y we get $y=\frac{60}{37}$. Thus $\frac{x}{y}=(\mathrm{D})\frac{37}{35}$.

22 Sides $\overline{AB}$ and $\overline{AC}$ of equilateral triangle ABC are tangent to a circle at points B and C respectively. What fraction of the area of $\triangle ABC$ lies outside the circle? (2017 AMC 10A Problem, Question#22)

A. $\frac{4\sqrt{3}\pi}{27}-\frac{1}{3}$
B. $\frac{\sqrt{3}}{2}-\frac{\pi}{8}$
C. $\frac{1}{2}$
D. $\sqrt{3}-\frac{2\sqrt{3}\pi}{9}$
E. $\frac{4}{3}-\frac{4\sqrt{3}\pi}{27}$

Answer E

Solution



Let the radius of the circle be r, and let its center be O.

Since $\overline{AB}$ and $\overline{AC}$ are tangent to circle O, then $\angle OBA=\angle OCA=90°$, so $\angle BOC=120°$. Therefore, since $\overline{OB}$ and $\overline{OC}$ are equal to r, then (pick your favorite method) $\overline{BC}=r\sqrt{3}$. The area of the equilateral triangle is $\frac{(r\sqrt{3})^2\sqrt{3}}{4}=\frac{3r^2\sqrt{3}}{4}$ and the area of the sector we are subtracting from it is $\frac{1}{3}\pi r^2-\frac{1}{2}r\cdot r\cdot\frac{\sqrt{3}}{2}=\frac{\pi r^2}{3}-\frac{r^2\sqrt{3}}{4}$. The area outside of the circle $\frac{3r^2\sqrt{3}}{4}-(\frac{\pi r^2}{3}-\frac{r^2\sqrt{3}}{4})=r^2\sqrt{3}-\frac{\pi r^2}{3}$. Therefore, the answer is

$$\frac{r^2\sqrt{3}-\frac{\pi r^2}{3}}{\frac{3r^2\sqrt{3}}{4}}=(\mathrm{E})\frac{4}{3}-\frac{4\sqrt{3}\pi}{27}.$$

23 How many triangles with positive area have all their vertices at points (i,j) in the coordinate plane, where i and j are integers between 1 and 5, inclusive? (2017 AMC 10A Problem, Question#23)

A. 2128 B. 2148 C. 2160 D. 2200 E. 2300

Answer B

Solution We can solve this by finding all the combinations, then subtracting the ones that are on the same line. There are 25 points in all, from $(1,1)$ to $(5,5)$, so $\binom{25}{3}$ is $\frac{25\cdot 24\cdot 23}{3\cdot 2\cdot 1}$, which simplifies to 2300.Now we count the ones that are on the same line. We see that any three points chosen from $(1,1)$ and $(1,4)$ would be on the same line, so $\binom{5}{3}$ is 10, and there are 5 rows, 5 columns,

and 2 long diagonals, so that results in 120. We can also count the ones with 4 on a diagonal. That is $\binom{4}{3}$, which is 4 and there are 4 of those diagonals, so that results in 16. We can count the ones with only 3 on a diagonal, and there are 4 diagonals like that, so that results in 4. We can also count the ones with a slope of $\frac{1}{2}$, 2, $-\frac{1}{2}$, or-2, with 3 points in each. There are 12 of them, so that results in 12. Finally, we subtract all the ones in a line from 2300, so we have $2300-120-16-4-12=(\mathrm{B})2148$.

24 For certain real numbers a, b, and c, the polynomial $g(x)=x^3+ax^2+x+10$ has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$f(x)=x^4+x^3+bx^2+100x+c$. What is $f(1)$? (2017 AMC 10A Problem, Question#24)

A. -9009 B. -8008 C. -7007 D. -6006 E. -5005

Answer C

Solution $f(x)$ must have four roots, three of which are roots of $g(x)$. Using the fact that every polynomial has a unique factorization into its roots, and since the leading coefficient of $f(x)$ and $g(x)$ are the same, we know that $f(x)=g(x)(x-r)$

where $r\in C$ is the fourth root of $f(x)$. Substituting $g(x)$ and expanding, we find that

$f(x)=(x^3+ax^2+x+10)(x-r)$

$=x^4+(a-r)x^3+(1-ar)x^2+(10-r)x-10r$. Comparing coefficients with $f(x)$, we see that

$a-r=1$,

$1 - ar = b$,

$10 - r = 100$,

$-10r = c$.

(Solution 1. 1 picks up here.)

Let's solve for a, b, c, and r. Since $10 - r = 100$, $r = -90$, so $c = (-10)(-90) = 900$. Since $a - r = 1$, $a = -89$, and $b = 1 - ar = -8009$. Thus, we know that

$f(x) = x^4 + x^3 - 8009x^2 + 100x + 900$.

Taking $f(1)$, we find that

$f(1) = 1^4 + 1^3 - 8009(1)^2 + 100(1) + 900$,

$= 1 + 1 - 8009 + 100 + 900$,

$= (\mathrm{C}) - 7007$.

A faster ending to Solution 1 is as follows. We shall solve for only a and r.

Since $10 - r = 100$, $r = -90$, and since $a - r = 1$, $a = -89$.Then,

$f(1) = (1 - r)(1^3 + a \cdot 1^2 + 1 + 10)$,

$= (91)(-77)$,

$= (\mathrm{C}) - 7007$.

We notice that the constant term of $f(x) = c$ and the constant term in $g(x) = 10$.

Because $f(x)$ can be factored as $g(x) \cdot (x - r)$ (where ris the unshared root of $f(x)$, we see that using the constant

Nowwe once agan wile $f(x)$ out in factored fom term, $-10 \cdot r = c$ and therefore $r = -\frac{c}{10}$. Now we once again write $f(x)$ out in factored form:

$f(x) = g(x) \cdot (x - r) = (x^3 + ax^2 + x + 10)(x + \frac{c}{10})$.

We can expand the expression on the right–hand side to get:

$f(x) = x^4 + (a + \frac{c}{10})x^3 + (1 + \frac{ac}{10})x^2 + (10 + \frac{c}{10})x + c$

Now we have

$f(x) = x^4 + (a + \frac{c}{10})x^3 + (1 + \frac{ac}{10})x^2 + \left(10 + \frac{c}{10}\right)x + c = x^4 + x^3 + bx^2 + 100x + c$.

Simply looking at the coefficients for each corresponding term (knowing that they must be equal), we have the equations

$10 + \frac{c}{10} = 100 \Rightarrow c = 900 a + \frac{c}{10} = 1$, $c = 900 \Rightarrow a + 90 = 1 \Rightarrow a = -89$ and finally,

$1 + \frac{ac}{10} = b = 1 + \frac{-89 \cdot 900}{10} = b = -8009$.

We know that $f(1)$ is the sum of its coefficients, hence $1+1+b+100+c$. We substitute the values we obtained for b and c into this expression to get

$f(1)=1+1+(-8009)+100+900=(\mathrm{C})-7007$.

Let r_1, r_2, and r_3 be the roots of $g(x)$, Let r^4 be the additional root of $f(x)$.Then from Vieta's formulas on the quadratic term of $g(x)$ and the cubic term of $f(x)$, we obtain the following:

$r_1+r-2+r_3=-a$,

$r_1+r_2+r_3+r_4=-1$,

Thus $r_4=a-1$.

Now applying Vieta's formulas on the constant term of $g(x)$, the linear term of $g(x)$, and the linear term of $f(x)$, we obtain:

$r_1r_2r_3=-10$.

$r_1r_2+r_2r_3+r_3r_1=1$.

$r_1r_2r3+r_2r_3r_4+r_3r_4r_1+r_4r_1r_2=-100$.

Substituting for $r_1r_2r_3$ in the bottom equation and factoring the remainder of the expression, we obtain:

$-10+(r_1r_2+r_2r_3+r_3r_1)r_4=-10+r_4=-100$.

It follows that $r_4=-90$. But $r_4=a-1$ so $a=-89$.

Now we can factor $f(x)$ in terms of $g(x)$ as $f(x)=(x-r_4)g(x)=(x+90)g(x)$.

Then $f(1)=91g(1)$and $g(1)=1^3-89\cdot1^2+1+10=-77$.

Hence $f(1)=91\cdot(-77)=(\mathrm{C})-7007$.

Let the roots of $g(x)$ be r_1, r_2, and r_3. Let the roots of $f(x)$ be r_1, r_2, r_3, and r_4. From Vieta's, we

$r_1+r_2+r_3=-a$,

$r_1+r_2+r_3+r_4=-1$.

have: $r_4=a-1$ The fourth root is $a-1$. Since r_1, r_2, and r_3 are common roots, we

$f(x)=g(x)(x-(a-1))$.

$f(1)=g(1)(1-(a-1))$.

$f(1)=(a+12)(2-a)$.

have: $f(1) = -(a+12)(a-2)$ Let $a-2=k$,$f(1) = -k(k+14)$Nole that $-7007 = -1001 \cdot (7) = -(7 \cdot (11) \cdot (13)) \cdot (7) = -91 \cdot (77)$This gives us a pretty good guess of $(\mathrm{C}) - 7007$.

First off, let's get rid of the x^4 term by finding $h(x) = f(x) - xg(x)$.This polynomial consists of the difference of two polynomials with 3 common factors, so it must also have these factors. The polynomial is $h(x) = (1-a)x^3 + (b-1)x^2 + 90x + c$, and must be equal to $(1-a)g(x)$.

Equating the coefficients, we get 3 equations. We will tackle the situation one equation at a time, starting the x tems. Looking at the coefficients, we get $\frac{90}{1-a} = 1$. $90 = 1-a$.The solution to the previous is obviously $a = -89$. We can now find b and c. $\frac{b-1}{1-a} = a$

$\therefore b-1 = a(1-a) = -89 * 90 = -8010$ and $b = -8009$. Fmaly $\frac{c}{1-a} = 10$,

$\therefore c = 10(1-a) = 10 * 90 = 900$ Solving the original problem .

$f(1) = 1+1+b+100+1 = 102+b+c = 102+900-8009 = (\mathrm{C}) - 7007$.

25 How many integers between 100 and 999, inclusive, have the property that some permutation of its digits is a multiple of 11 between 100 and 999? For example, both 121 and 211 have this property. (2017 AMC 10A Problem, Question#25)

A. 226 B. 243 C. 270 D. 469 E. 486

Answer A

Solution Let the three−digit number be ACB:

If a number is divisible by 11, then the difference between the sums of alternating digits is a multiple of 11.

There are two cases: $A+B=C$ and $A+B=C+11$ We now proceed to break down the cases. Note: let $A \geq C$ so that we avoid counting the same permutations and having to subtract them later.

Case 1: $A+B=C$.

Part 1: $B = 0$, $A = C$, this case results in 110, 220, $330 \ldots 990$. There are two ways to arrange the digits in each of those numbers. $2 \cdot 9 = 18$.

Part 2: $B = 1$, $A + 1 = C$, this case results in 121, 231, $\ldots 891$. There are 6 ways to arrange the digits in all of those number except the first, and 3 ways for the first. This leads to 45 cases.

Part 3: $B = 2$, $A + 2 = C$, this case results in 242, 352,$\ldots 792$. There are 6 ways to arrange the digits in all of those number except the first, and $\$3$ ways for the first. This leads to 33 cases.

Part 4: $B = 3$, $A + 3 = C$, this case results in 363, 473, $\ldots 693$. There are 6 ways to arrange the digits in all of those number except the first, and 3ways for the first. This leads to 21 cases.

Part 5: $B = 4$, $A + 4 = C$, this case results in 484 and 594. There are 6ways to arrange the digits in 594 and 3 ways for 484. This leads to 9 cases. This case has $18 + 45 + 33 + 21 + 9 = 126$ subcases.

Case 2: $A + B = C + 11$.

Part 1: $C = 0$, $A + B = 11$, this cases results in 209, 308, 407, 506. There are 4 ways to arrange each of those cases. This leads to 16 cases.

Part 2: $C = 1$, $A + B = 12$, this cases results in 319, 418, 517,616. There are 6 ways to arrange each of those cases, except the last. This leads to 21 cases.

Part 3: $C = 2$, $A + B = 13$, this cases results in 429, 528, 627. There are 6 ways to arrange each of those cases. This leads to 18 cases.

$\ldots$ If we continue this counting, we receive $16 + 21 + 18 + 15 + 12 + 9 + 6 + 3 = 100$ subcases. $100 + 126 = (\mathrm{A})226$.

We note that we only have to consider multiples of 11 and see how many valid permutations each has. We can do casework on the number of repeating digits that the multiple of 11 has:

Case 1: All three digits are the same. By inspection, we find that there are no multiples of 11 here.

Case 2: Two of the digits are the same, and the third is different.

Case $2a$: There are 8 multiples of 11 without a zero that have this property: 121, 242, 363, 484, 616, 737, 858, 979. Each contributes 3 valid permutations, so there are $8.3 = 24$ permutations in this subcase.

Case $2b$: There are 9 multiples of 11 with a zero that have this property: 110, 220, 330, 440, 550, 660, 770, 880, 990. Each one contributes 2 valid permutations (the first digit can't be zero) so there are $9 \cdot 2 = 18$ permutations in this subcase.

Case 3: All the digits are different Since there are $\frac{990-110}{11} + 1 = 81$ multiples of 11 between 100 and 999, there are $81 - 8 - 9 = 64$ multiples of 11 remaining in this case. However, 8 of them contain a zero, namely 209, 308, 407, 506, 605, 704, 803, and 902. Each of those multiples of 11 contributes $2 \cdot 2 = 4$ valid permutations, but we overcounted by a factor of 2; every permutation of 209, for example, is also a permutation of 902. Therefore, there are $8 \cdot 4/2 = 16$. Therefore, there are $64 - 8 = 56$ remaining multiples of 11 without a 0 in this case. Each one contributes $3! = 6$ valid permutations, but once again, we overcounted by a factor of 2 (note that if a number ABC is a muliple of 11, then so is CBA). Therefore, there are $56 \cdot 6/2 = 168$ valid permutations in this subcase. Adding up all the permutations from all the cases, we have $24 + 18 + 16 + 168 = (\mathrm{A})226$.

We can overcount and then subtract. We know there are 81 multiples of 11.

We can multiply by 6 for each permutation of these mutiples. (Yet some multiples don't have 6) Now divide by 2, because if a number abc with digits a, b, and c is a multiple of 11, then cba is also a multiple of 11 so we have counted the same permutations twice.

Basically, each multiple of 11 has its own 3 permutations (say abc has abc acb and bac whereas cba has cba cab and bca). We know that each multiple of 11 has at least 3 permutations because it cannot have 3 repeating digits.

Hence we have 243 permutations without subtracting for overcounting. Now note that we overcounted cases in which we have $0's$ at the start of each number. So, in theory, we could just answer A and move on.

If we want to solve it, then we continue.

We overcounted cases where the middle digit of the number is 0 and the last digit is 0. Note that we assigned each multiple of 113 permutations.

The last digit is 0 gives 9 possibilties where we overcounted by 1 permutation for each of $110, 220, \ldots, 990$.

The middle digit is 0 gives 8 possibilities where we overcount by 1. $605, 704, 803, 902$ and $506, 407, 308, 209$ subtracting 17 gives $(\mathrm{A})226$.

Now, we may ask if there is further overlap (I.e if two of abc and bac and acb were multiples of 11). Thankfully, using divisibility rules, this can never happen as taking the divisibility rule mod 11 and adding we get that $2a$, $2b$, or $2c$ is congruent to $0 (\mathrm{mod} 11)$. Since a, b, c are digits, this can never happen as none of them can equal 11 and they can't equal 0 as they are the leading digit of a 3 digit number in each of the cases.

As said in solution 3, each number only has at most 3 permutations. There are 81 multiples of 11, so the answer is at most 243 or B. But we know that we overcounted, so the answer is less than 243, leaving our only choice as A