# AMC 8 Number Theory

## AMC 8 Day 1 Odd & Even Number

1 The average of two odd numbers is always a _____ number.

A. odd B. even C. prime D. whole

Answer D

Solution Avg of $1$ & $5$ is $3$. Avg of $1$ and $7$ is $4$. Both are whole numbers.

So the answer is D.

2 If $n$ and $m$ are integers and $n^3+m^3$ is even, which of the following is impossible? (Adapted from 2014 AMC 8 Problem, Question #13)

A. $n$ and $m$ are even

B. $n$ and $m$ are odd

C. $n+m$ is even

D. $n+m$ is odd

E. none of these are impossible

Answer D

Solution Since $n^3+m^3$ is even, either both $n^3$ and $m^3$ are even, or they are both odd. Therefore, $n$ and $m$ are either both even or both odd, since the square of an even number is even and the square of an odd number is odd. As a result, $n+m$ must be even. The answer, then, is $\boxed{\mathrm{D}}$.

3 Suppose $m$ and $n$ are positive even integers. Which of the following must be an odd integer? (Adapted from 2005 AMC 8 Problem, Question #8)

A. $m+2n$ B. $3m-n$ C. $3m^2+3n^2$ D. $(nm+3)^2$ E. $3mn$

Answer D

Solution Only choice D can bring an odd base. Odd times odd number will always be an odd number.

4 If $n$ is an integer, then ___?___ must be even.

A. $n+1$

B. $n+2$

C. $2\times n+1$

D. $2\times n+2$

E. $3\times n$

Answer D

Solution $\mathrm{A}$ and $\mathrm{C}$ are odd if $n=2$; $\mathrm{B}$ is odd if $n=1$.

So the answer is D.

5 Let $o$ be an even whole number and let $n$ be any whole number. Which of the following statements about the whole number $(1+o\times n)^2$ is always true? (Adapeted from 1986 AJHSME Problem, Question #17)

A. it is always odd

B. it is always even

C. it is even only if $n$ is even

D. it is odd only if $n$ is odd

E. it is odd only if $n$ is even

Answer A

Solution No matter what $n$ is, the product of $on$ must be even.

The base $1+on$ must be odd, thus the answer is an odd number.

## AMC 8 Day 2 Divisibility

1. A four digit number $\overline{A34B}$ is divisible by $5$ and $9$. $A+B$ is ______ .

A. $7$ B. $11$ C. $2$ D. $2$ or $9$ E. $2$ or $11$

Answer E

Solution For this number to be divisible by $5$, $B$ is either $0$ or $5$.

For this number to be divisible by $9$:

When $B=0$, $A+3+4+0=7+A$ needs to be divisible by $9$, so $A=2$. The number is $2340$.

When $B=5$, $A+3+4+5=12+A$needs to be divisible by $9$, so $A=6$. The number is $6345$.

Therefore, the number is $2340$ or $6345$.

2. The number $\overline{2A6A}$ is divisible by $5$ and $9$. This four digit number is ______ .

Answer $2565$

Solution For this number to be divisible by $5$, $A$ can only be $0$ or $5$.

For this number to be divisible by $9$:

When $A=0$, $2+0+6+0=8$ can't be divided by $9$.

When $A=5$, $2+5+6+5=18$ can be divided by $9$. So, this four digit number is $2565$.

3. How many different digits can be filled in the square to make the number be divisible by $3$?

$$\overline{135\square8}$$

A. $0$ B. $1$ C. $2$ D. $3$ E. $4$

Answer D

Solution The sum of the digits need to be a multiple of $3$. $1+3+5+8=17$, which means the digit in the square can only be $1,4$ or $7$.

4 How many integers between $1$ and $2013$ (inclusive) are divisible by $6$ or $8$?

A. $301$ B. $503$ C. $1006$ D. $500$ E. $48$

Answer B

Solution $\lfloor\frac{2013}{6}\rfloor+\lfloor\frac{2013}{8}\rfloor-\lfloor\frac{2013}{24}\rfloor=503$, so there are $503$ integers between $1$ and $2013$ that are divisible by $6$ or $8$.

5 The sum of all digits is a multiple of $9$ for some three digit numbers. How many three digit numbers like this are there?

A. $111$ B. $267$ C. $100$ D. $110$

Answer C

Solution $12\times9=108$, $111\times9=999$.

$(999-108)\div9+1=891\div9+1=99+1=100$.

## AMC 8 Day 3 Prime Numbers and Composite Numbers

1 "If a whole number $n$ is not prime, then the whole number $n-2$ is not prime." A value of $n$ which shows this statement to be false is ______ . (1987 AMC 8 Problem, Question#20)

A. $9$ B. $12$ C. $13$ D. $16$ E. $23$

Answer A

Solution To show this statement to be false, we need a non-prime value of $n$ such that $n-2$ is prime. Since $13$ and $23$ are prime, they won't prove anything relating to the truth of the statement. Now we just check the statement for $n=9, 12, 16$. If $n=12$ or $n=16$, then $n-2$ is $10$ or $14$, which aren't prime. However, $n=9$ makes $n-2=7$, which is prime, so $n=9$ proves the statement false.

2 The sum of two prime numbers is $81$. What is the product of these two prime numbers? (Adapted from 2014 AMC 8 problem, Question#4)

A. $85$ B. $91$ C. $115$ D. $133$ E. $158$

Answer E

Solution Since the two prime numbers sum to an odd number, one of them must be even. The only even prime number is $2$. The other prime number is $81-2=79$, and the product of these two numbers is $79 \cdot 2=158$.

3 What is the sum of the distinct prime divisors of $2022$? (Adaped from 2016 AMC 8 Problem, Question#9)

A. $341$ B. $342$ C. $344$ D. $345$ E. $347$

Answer B

Solution The prime factorization is $2022=2\times 3\times 337$. Since the problem is only asking us for the distinct prime factors, we have $2, 3, 337$. Their desired sum is then $342$.

4 If $m$ is prime and $n$ is composite, which of the following must be composite?

A. $m+n$ B. $n+2m$ C. $2(n-m)$ D. $2(n+m)$ E. $m-n$

Answer D

Solution $\mathrm{A}. \times m=3$ , $n=4$ , $m+n=3+4=7$;

$\mathrm{B}. \times m=2$ , $n=9$ , $n+2m=9+2\times2=13$;

$\mathrm{C}. \times m=3$ , $n=4$ , $2(n-m)=2\times(4-3)=2$;

$\mathrm{D}$. This number is an even number greater than $4$.

$\mathrm{E}. \times m=17$, $n=8$, $m-n=17-8=9$.

5 In how many ways can $10001$ be written as the sum of two primes?

A. $0$ B. $1$ C. $2$ D. $3$ E. $4$

Answer A

Solution For the sum of two numbers to be odd, one must be odd and the other must be even, because All odd numbers are of the form $2n+1$ where $n$ is an integer, and all even numbers are of the form $2m$ where $m$ is an integer.

$2n+1+2m=2m+2n+1=2(m+n)+1$ and $m+n$ is an integer because $m$ and $n$ are both integers. The only even prime number is $2$, so our only combination could be $2$ and $9999$. However, $9999$ is clealy divisible by $3$ so the number of ways $10001$ can be written as the sum of two primes is $\boxed{(\mathrm{A})0}$.

## AMC 8 Day 4 GCF & LCM

1 Abe, Bob, and Carl go to the library. Abe goes every $3$ days, Bob goes every $4$ days, and Carl goes every $2$ days. If they meet at the library on April $25^{th}$, when will they meet again?

A. May $6^{th}$ B. May $7^{th}$ C. May $18^{th}$ D. May $19^{th}$

Answer B

Solution The least common multiple of $2$, $3$, $4$ is $12$, so they will meet again in $12$ days, which is May $7^{th}$.

2 Wanda, Darren, Beatrice, and Chi are tutors in the school math lab. Their schedules are as follows: Darren works every second school day, Wanda works every fourth school day, Beatrice works every sixth school day, and Chi works every seventh school day. Today they are all working in the math lab. In how many school days from today will they next be together tutoring in the lab? (Adapted from 2001 AMC10 Problem, Question#8)

A. $42$ B. $84$ C. $126$ D. $178$ E. $252$

Answer B

Solution We need to find the least common multiple of the four numbers given. That is, the next time they will be together. First, find the least common multiple of $2$ and $4$, which is $4$.

Find the least common multiple of $4$ and $6$, and the least common multiple is $12$.

Lastly, the least common multiple of $12$ and $7$ is $12 \times 7 = 84$.

3 Each piece of candy in a store costs a whole number of cents. Casper has exactly enough money to buy either $12$ pieces of red candy, $14$ pieces of green candy, $15$ pieces of blue candy, or $n$ pieces of purple candy. A piece of purple candy costs $21$ cents. What is the smallest possible value of $n$?(Adapted from 2019 AMC 10B Problem, Question#7)

A. $18$ B. $20$ C. $24$ D. $25$ E. $28$

Answer B

Solution If he has enough money to buy $12$ pieces of red candy, $14$ pieces of green candy, and $15$ pieces of blue candy, then the smallest amount of money he could have is $\operatorname{lcm}(12, 14, 15) = 420$ cents. Since a piece of purpe candy costs $21$ cems, the smallest possible value of $n$ is $\dfrac{420}{21} = (\text{B})20$.

4 The least common divisor of $x$ and $y$ is $330$. The least common divisor of $y$ and $z$ is $180$. The least common divisor of $x$ and $z$ is $396$. Then $z=$ ______ .

A. $36$ B. $24$ C. $18$ D. $20$ E. $48$

Answer A

Solution $330=2\times3\times5\times11$, $180=2^2\times3^2\times5$, $396=2^2\times3^2\times11$. So $x$ must have a factor of $11$, and $y$ must have a factor of $5$, and $z$ must not have $11$ nor $5$ as its divisor. Also, the power of the prime factors $2$ and $3$ of $x$ and $y$ must be less than $2$. So, the prime factors of $z$ must have $2$ of $2$ and $2$ of $3$. So $z=2^2\times3^2=36$.

5 One day after the heavy snowfall, Tom and his father walked together to measure the perimeter of a circular flowerbed. Their starting point and walking direction were exactly the same. Each step of Tom is $54$ cm long, and his father's step is $72$ cm long. Since the footprints of them sometimes overlapped, they left $60$ footprints on the snow after each completed a circle. Then the circumference of the circular flowerbed is ______ cm.

A. $2160$ B. $3600$ C. $7200$ D. $1080$ E. $720$

Answer A

Solution We need to find out the distance traveled by two adjacent overlapping footprints and the number of times the footprints overlap in the entire distance. The distance traveled by the two people from the starting point to the first time their footprints overlap is the same, which is the least common multiple of their step lengths: $[54,72]=216$cm. In this $216$cm, they each left $216\div54=4$ and $216\div72=3$ footprints. Since they have one overlapping footprint, there are only $4+3-1=6$ footprints. $60\div6=10$, which means that they walked the around the flowerbed $10$ times. So, the circumference is $216\times10=2160$cm.

# AMC 8 Day 5 Number & Cycle

1 A $2$-digit number is such that the product of the digits plus the sum of the digits is equal to the number. What is the units digit of the number? (2014 AMC 8 Problem, Question #22)

A. $1$ B. $3$ C. $5$ D. $7$ E. $9$

Answer E

Solution We can think of the number as $10a+b$, where $a$ and $b$ are digits. Since the number is equal to the product of the digits $(a\cdot b)$ plus the sum of the digits $(a+b)$, we can say that $10a+b=a\cdot b+a+b$. We can simplify this to $10a=a\cdot b+a$, and factor to $(10)a=(b+1)a$. Dividing by $a$. We have that $b+1=10$. Therefore, the units digit, $b$, is $\boxed{(\mathrm{E})9}$.

2 A three-digit number's tens digit is $0$ and its value is $34$ times the sum of its digits. If we swap its ones digit and hundreds digit, the new number we get is ______ times the sum of its digits.

A. $67$ B. $34$ C. $3$ D. $17$ E. $201$

Answer A

Solution $\overline{a0b}=34(a+b)\Rightarrow 100a+b=34a+34b\Rightarrow 66a=33b\Rightarrow 2a=b$ ,

$$\frac{\overline{b0a}}{a+b}=\frac{100b+a}{a+b}=\frac{200a+a}{a+2a}=\frac{201a}{3a}=67.$$

3 The last digit of $2013^{2013}$ is ______ .

A. $1$ B. $2$ C. $3$ D. $4$

Answer C

Solution The last digits of the exponents to the power of $2013$ can form the cycle of $3$, $9$, $7$, and $1$.

$2013\div 4=503R1$

The last digit of $3^{2013}$ is equal to the last digit of $3^1$, which is $3$ .

4 The remainder of $4^{2019}$ divided by $5$ is?

A. $0$ B. $1$ C. $2$ D. $3$ E. $4$

Answer E

Solution $4^1 \div 5 = 0R4$

$4^2 \div 5 = 3R1$

$4^3 \div 5 = 12R4$

$4^4 \div 5 = 51R1$

The remainder should be $4$ or $1$.

Because $2019$ is an odd exponent, the remainder should be $4$.

5 Today is Sunday. What day of the week is it after $40^{40}$ days? ______ .

A. Monday B. Tuesday C. Wednesday D. Friday E. Sunday

Answer B

Solution $40^{40} \equiv 5^{40} (\bmod 7)$ ,

$5^{40} = 25^{20} \equiv 4^{20} (\bmod 7)$ ,

$4^{20} = 16^{10} \equiv 2^{10} (\bmod 7)$ ,

$2^{10} = 8^3 \times 2 \equiv 1^3 \times 2 (\bmod 7)$

Thus, $40^{40} \div 7R\ 2$.

So, the answer is Tuesday.

## AMC 8 Day 6 Chinese Remainder Theorem

1 The remainder of a three-digit number is $1$ when it is divided by $3$ or $4$ or $5$. What is the least value of such a number? ______ .

Answer $121$

Solution $[3,4,5]+1=61$ ,

$61+60=121$ .

2 The remainder of a three-digit number is always $2$ when it is divided by $3$, $5$, or $7$. What is the sum of digit of the largest value of such a number? ______ .

A. $20$ B. $18$ C. $16$ D. $14$ E. $12$

Answer A

Solution $[3,5,7]=105$ , $105\times 9+2=947$ .

3 The remainder is $2$ when a number is divided by $3$, is $3$ when it is divided by $4$, is $4$ when it is divided by $5$. The smallest value of such a number is ______ , the largest value of such a three-digit number is ______ .

Answer 1:$59$

2:$959$

Solution After adding $1$, the number should be divisible by $3$, $4$, and $5$.

The smallest number should be $[3,4,5]-1=59$.

4 The remainder of a number is $2$ when it is divided by $3$ and is $3$ when divided by $7$. What is the smallest value of such a number?

A. $32$ B. $96$ C. $17$ D. $58$ E. $59$

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Answer C

Solution Check from $2$, each time add $3$ more.

$2 \div 7 = 0R2$

$5 \div 7 = 0R5$

$8 \div 7 = 1R1$

$11 \div 7 = 1R4$

$14 \div 7 = 2$

$17 \div 7 = 2R3$

5 The remainder of a natural number is $4$ when it is divided by $5$, is $4$ when divided by $6$, and is $7$ when divided by $9$. What is the sum of digit of the smallest value of such a number? ______ .

A. $3$ B. $4$ C. $5$ D. $6$ E. $7$

Answer E

Solution After minus $4$, the number is a multiple of both $5$ and $6$, which means the number should be $4$ at least, then $34$, $64$, $94\cdots$

After trying, $34 \div 9 = 3R7$.

The smallest number is $34$, and the answer is $3 + 4 = 7$.