2025 AMC 10A

1 Andy and Betsy both live in Mathville. Andy leaves Mathyille on his bicycle at $1:30$, traveling due north at a steady 8 miles per hour. Betsy leaves on her bicycle from the same point at $2:30$, traveling due east at a steady 12 miles per hour. At what time will they be exactly the same distance from their common starting point?

A. $3:30$ B. $3:45$ C. $4:00$ D. $4:15$ E. $4:30$

Answer E

Solution Let x represent Betsy's travel time in hours. Then Andy's travel time is $x+1$ hours. Setting up the equation based on distance traveled: $8(x+1)=12x$, which yields $x=2$. Therefore, the meeting time is 4:30 PM.

2 A box contains 10 pounds of a nut mix that is 50 percent peanuts, 20 percent cashews, and 30 percent almonds. A second nut mix containing 20 percent peanuts, 40 percent cashews, and 40 percent almonds is added to the box resulting in a new nut mix that is 40 percent peanuts. How many pounds of cashews are now in the box?

A. 3.5 B. 4 C. 4.5 D. 5 E. 6

Answer B

Solution Define x as the weight (in pounds) of the second mixture.

The proportion of peanuts in the combined mixture satisfies:

$\frac{0.2x+10\times 0.5}{x+10}=0.4 \Rightarrow 5+0.2x=4+0.4x, \quad 0.2x=1, \quad x=5$. Therefore, the cashew content equals $10\times 0.2+5\times 0.4=4$ pounds.

3

How many isosceles triangles are there with positive area whose side lengths are all positive integers and whose longest side has length 2025?

A. 2025 B. 2026 C. 3012 D. 3037 E. 4050

Answer D

Solution Consider isosceles triangles with side lengths a, a, b where $a, b \in \mathbb{N}_+$.

Scenario 1: $b = 2025$.

By triangle inequality: $a + a > b$, which gives $2a > 2025$, so $a \geq 1013$.

Therefore, $1013 \leq a \leq 2025$ gives us 1013 valid triangles.

Scenario 2: $a = 2025$.

We have $1 \leq b \leq 2025$, yielding 2025 valid triangles.

Notice that we count the equilateral with side length 2025 twice, so the combined total is $1013 + 2025 - 1 = 3037$ triangles.

4 A team of students is going to compete against a team of teachers in a trivia contest. The total number of students and teachers is 15. Ash, a cousin of one of the students, wants to join the contest. If Ash plays with the students, the average age on that team will increase from 12 to 14. If Ash plays with the teachers, the average age on that team will decrease from 55 to 52. How old is Ash?

A. 28 B. 29 C. 30 D. 32 E. 33

Answer A

Solution Let n denote the number of students, making $(15 - n)$ the number of teachers. Let a represent Ash's age.

From the students' average age: $12n + a = (n + 1) \times 14$

From the teachers' average age: $55 \cdot (15 - n) + a = 52 \cdot (15 - n + 1)$

Solving this system: - From the first equation: $a = 2n + 14$ - From the second equation: $a = 3n + 7$

Therefore $2n + 14 = 3n + 7$, which gives $n = 7$ and $a = 28$.

5 Consider the sequence of positive integers

$$1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1, 2, \cdots$$

What is the 2025th term in this sequence?

A. 5 B. 15 C. 16 D. 44 E. 45

Answer E

Solution Divide the whole sequence at each 1.

The n-th sequence would be: $1, 2, \ldots, n+1, \ldots, 2$, which has length $2n$.

The cumulative length until the end of the n-th sequence is:

$$f(n) = \sum_{k=1}^{n} 2k = 2 \cdot \frac{n(n+1)}{2} = n(n+1)$$

Computing: $f(44) = 44 \times 45 = 1980$ and $f(45) = 45 \times 46 = 2070$.

Hence, the 2025th term appears at position $(2025 - 1980) = 45$ within the 45th sequence

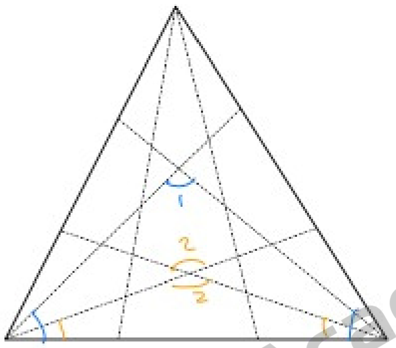
$1, 2, \ldots, 46, \ldots, 2$, giving us 45.

6 In an equilateral triangle each interior angle is trisected by a pair of rays. The intersection of the interiors of the middle $20°$-angle at each vertex is the interior of a convex hexagon. What is the degree measure of the smallest angle of this hexagon?

A. 80 B. 90 C. 100 D. 110 E. 120

Answer C

Solution



Computing the angles: $\angle 1 = 180^\circ - 40^\circ - 40^\circ = 100^\circ$ $\angle 2 = 180^\circ - 20^\circ - 20^\circ = 140^\circ$

By symmetry, the hexagon's six angles are $100^\circ, 100^\circ, 100^\circ, 140^\circ, 140^\circ, 140^\circ$.

7 Suppose a and b are real numbers. When the polynomial $x^3 + x^2 + ax + b$ is divided by $x - 1$, the remainder is 4. When the polynomial is divided by $x - 2$, the remainder is 6. What is $b - a$?

A. 14 B. 15 C. 16 D. 17 E. 18

Answer E

Solution Define $f(x) = x^3 + x^2 + ax + b$.

Using the given conditions: - $f(1) = 1 + 1 + a + b = 4$ - $f(2) = 8 + 4 + 2a + b = 6$

This gives us: - $a + b = 2$ - $2a + b = -6$

Solving: $a = -8$ and $b = 10$.

Consequently, $b - a = 10 - (-8) = 18$.

8 Agnes writes the following four statements on a blank piece of paper.

· At least one of these statements is true.

· At least two of these statements are true.

· At least two of these statements are false.

· At least one of these statements is false.

Each statement is either true or false. How many false statements did Agnes write on the paper?

A. 0 B. 1 C. 2 D. 3 E. 4

Answer B

Solution Denote the four statements as s_1, s_2, s_3, s_4. Let T and F represent the counts of true and false statements respectively. We know $T + F = 4$.

Assume s_1 is false. This implies $T = 0, F = 4$, which would make s_4 false—a contradiction!

Since at least one of s_1, s_3 (or pick s_4) is true, at least one of s_2, s_4 (or pick s_3) is true, we have $T \geq 2$, confirming s_1, s_2 are true.

If s_4 were false, then $T = 4, F = 0$, contradicting s_4's falsity!

Hence, s_4 is true, yielding $T \geq 3, F \leq 1$, which implies s_3 is false, with $T = 3, F = 1$.

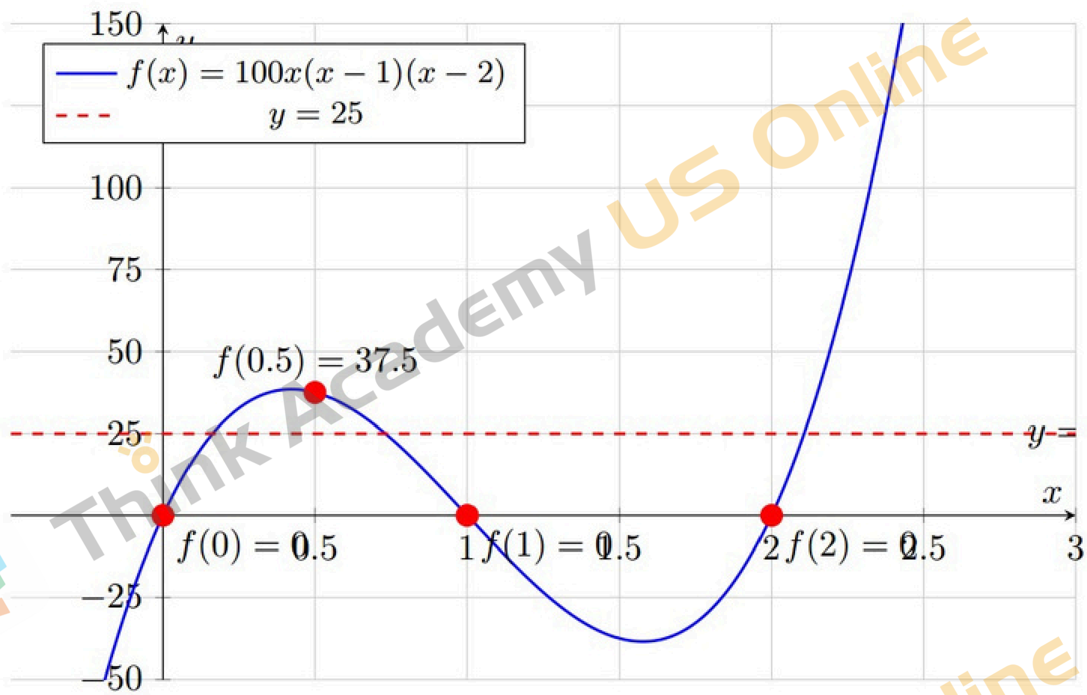
9 Let $f(x) = 100x^3 - 300x^2 + 200x$. For how many real numbers a does the graph of $y = f(x - a)$ pass through the point $(1, 25)$?

A. 1 B. 2 C. 3 D. 4

E. more than 4

Answer C

Solution



We seek the number of solutions of $f(1-a)=25$, which is equivalent to the number of solutions of $f(x)=25$.

Since $f(x)=100\times x(x-1)(x-2)$, we have $f(0)=f(1)=f(2)=0$. As the coefficient of x^3 is positive, we can draw the diagram of $f(x)$.

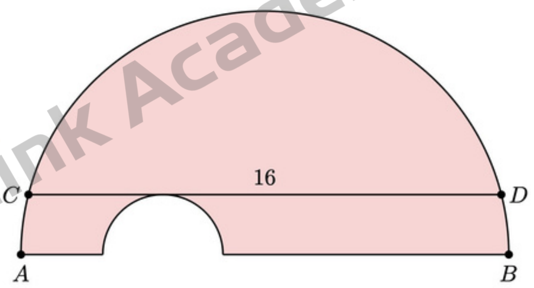
Note that $f\left(\frac{1}{2}\right)=100\times\frac{1}{2}\times\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)=25\times\frac{3}{2}>25$

By the intermediate value theorem, there exist $x_1\in\left(0,\frac{1}{2}\right)$ and $x_2\in\left(\frac{1}{2},1\right)$ satisfying $f(x_1)=f(x_2)=25$.

Since $f(2)=0$ and $f(3)>100>25$, there exists $x_3\in(2,3)$ with $f(x_3)=25$.

As $f(x)=25$ is a cubic equation with at most 3 real roots, x_1, x_2, x_3 are all the solutions.

10 A semicircle has diameter $\overline{AB}$ and chord $\overline{CD}$ of length 16 parallel to $\overline{AB}$. A smaller semicircle with diameter on $\overline{AB}$ and tangent to $\overline{CD}$ is cut from the larger semicircle, as shown below.

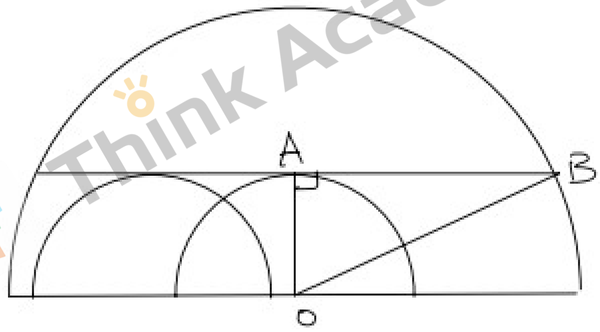


What is the area of the resulting figure, shown shaded?

A. 16π B. 24π C. 32π D. 48π E. 64π

Answer C

Solution



Let r and R denote the radii of the smaller and larger semicircles respectively.

As in the diagram, we can translate the tangent smaller semi-circle to be centered at O. So $OB = R$, $OA = r$, and $AB = 16/2 = 8$. Thus $R^2 - r^2 = 8^2$.

The answer is: $\frac{1}{2}\pi R^2 - \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(R^2 - r^2) = \frac{1}{2}\pi \times 64 = 32\pi$

11 The sequence $1, x, y, z$ is arithmetic. The sequence $1, p, q, z$ is geometric. Both sequences are strictly increasing and contain only integers, and z is as small as possible. What is the value of $x + y + z + p + q$?

A. 66 B. 91 C. 103 D. 132 E. 149

Answer E

Solution Let d be the common difference and r be the common ratio, where $d, r \in \mathbb{N}_+$ with $d, r > 1$.

The equation $1 + 3d = r^3$ holds. Since $r^3 \equiv 1 \pmod 3$, we have $r \equiv 1 \pmod 3$, implying $r \geq 4$.

For $r = 4$: $d = 21$, yielding $x = 22, y = 43, z = 64, p = 4, q = 16$.

$x + y + z + p + q = 22 + 43 + 64 + 4 + 16 = 149$

12 Carlos uses a 4-digit passcode to unlock his computer. In his passcode, exactly one digit is even, exactly one (possibly different) digit is prime, and no digit is 0. How many 4-digit passcodes satisfy these conditions?

A. 176 B. 192 C. 432 D. 464 E. 608

Answer D

Solution Scenario 1: The digit 2 serves as both prime and even. The remaining digits must be odd and not prime, thus they are in $\{1, 9\}$.

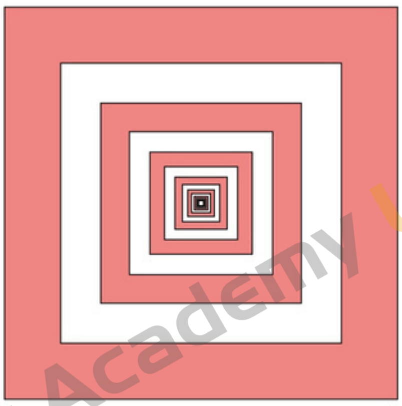
With 4 positions for 2, the count is $4 \times 2 \times 2 \times 2 = 32$.

Scenario 2: Excluding 2, the prime comes from $\{3, 5, 7\}$, the even digit from $\{4, 6, 8\}$, and others from $\{1, 9\}$. There are $4 \times 3 = 12$ arrangements for prime and even positions.

The count is $12 \times 3 \times 3 \times 2 \times 2 = 432$.

Total count: $32 + 432 = 464$.

13 In the figure below, the outside square contains infinitely many squares, each of them with the same center and sides parallel to the outside square. The ratio of the side length of a square to the side length of the next inner square is k, where $0 < k < 1$. The spaces between squares are alternately shaded, as shown in the figure (which is not necessarily drawn to scale).



The area of the shaded portion of the figure is 64% of the area of the original square. What is k?

A. $\frac{3}{5}$ B. $\frac{16}{25}$

C. $\frac{2}{3}$

D. $\frac{3}{4}$

E. $\frac{4}{5}$

Answer D

Solution Solution 1:

Notice that, the shaded area and the non-shaded area are also similar, with the length ratio $1:k$, and therefore an area ratio $1:k^2$.

Thus the shaded area is $\frac{1}{1+k^2}=64\%$, giving $k=\frac{3}{4}$.

Solution 2:

The total area forms an infinite geometric series:

$1-k^2+k^4-k^6+k^8-k^{10}+\cdots=\frac{1}{1-(-k^2)}=\frac{1}{1+k^2}$

Setting $\frac{1}{1+k^2}=0.64$ yields $k=\frac{3}{4}$.

14 Six chairs are arranged around a round table. Two students and two teachers randomly select four of the chairs to sit in. What is the probability that the two students will sit in two adjacent chairs and the two teachers will also sit in two adjacent chairs?

A. $\frac{1}{6}$

B. $\frac{1}{5}$

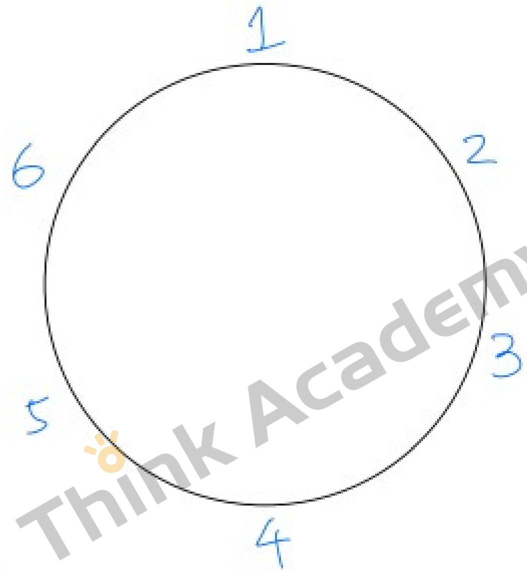
C. $\frac{2}{9}$

D. $\frac{3}{13}$

E. $\frac{1}{4}$

Answer B

Solution



We may label the six seats as $1, 2, 3, 4, 5, 6$ on a round table.

Choose the two seats for the two students (order of the two students does not matter).

There are $\binom{6}{2}$ unordered choices in total, while the adjacent pairs are $(1,2), (2,3), (3,4), (4,5), (5,6), (6,1)$, hence 6 choices. Therefore

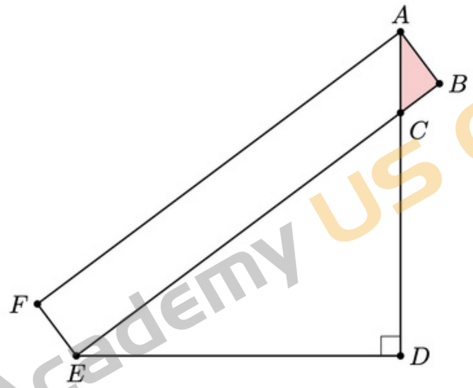
$P(\text{adjacent\ students}) = \frac{6}{\binom{6}{2}} = \frac{2}{5}$

Given that the students are adjacent, without loss of generality, suppose they occupy seats $(1,2)$. The remaining seats are $\{3,4,5,6\}$. Among these four, the adjacent pairs (on the circle) are $(3,4), (4,5), (5,6)$, so there are 3 favorable pairs for the two teachers out of $\binom{4}{2}$ unordered choices. Thus $P(\text{adjacent\ teachers} \mid \text{adjacent\ students}) = \frac{3}{\binom{4}{2}} = \frac{1}{2}$

Thus the final answer is

$P(\text{adjacent\ students}) \cdot P(\text{adjacent\ teachers} \mid \text{adjacent\ students}) = \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{5}$

15 In the figure below, $ABEF$ is a rectangle, $\overline{AD} \perp \overline{DE}$, $AF = 7$, $AB = 1$, and $AD = 5$



What is the area of $\triangle ABC$?

A. $\frac{3}{8}$

B. $\frac{4}{9}$

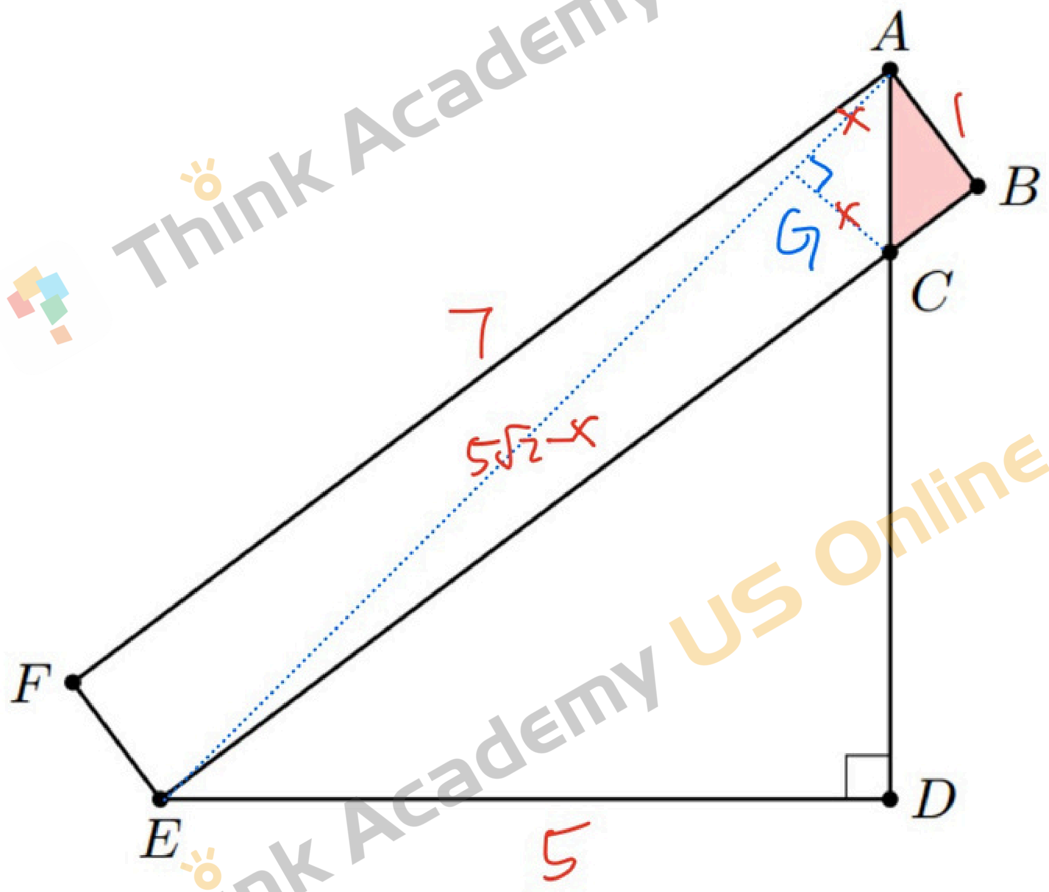
C. $\frac{1}{8}\sqrt{13}$

D. $\frac{7}{15}$

E. $\frac{1}{8}\sqrt{15}$

Answer A

Solution



By Pythagorean theorem, we can see that $AE = 5\sqrt{2}$, $DE = 5$. So $\triangle ADE$ is a right isoceles triangle, $\angle EAD = 45^\circ$.

Make altitude $CG \perp AE$ at G. Set $CG = AG = x$, then $EG = 5\sqrt{2} - x$. As $\triangle ECG \sim \triangle EAB$, $\frac{EG}{GC} = \frac{5\sqrt{2}-x}{x} = \frac{EB}{BA} = 7$

So $x = \frac{5\sqrt{2}}{8}$. Area calculation: $S_{\triangle ABC} = S_{\triangle ABE} - S_{\triangle ACE} = \frac{1}{2}AB \cdot BE - \frac{1}{2}AE \cdot CG = \frac{3}{8}$.

16 There are three jars. Each of three coins is placed in one of the three jars, chosen at random and independently of the placements of the other coins. What is the expected number of coins in a jar with the most coins?

A. $\frac{4}{3}$

B. $\frac{13}{9}$

C. $\frac{5}{3}$

D. $\frac{17}{9}$

E. 2

Answer D

Solution Total arrangements for placing 3 coins in 3 jars: $3^3 = 27$.

If maximum coins per jar = 1: Each jar contains exactly one coin. Count: $3! = 6$.

If maximum coins per jar = 3: One jar holds all coins. Count: 3.

Remaining arrangements ($27 - 6 - 3 = 18$): One jar has 2 coins, another has 1.

Expected maximum: $\frac{6}{27} \times 1 + \frac{18}{27} \times 2 + \frac{3}{27} \times 3 = \frac{17}{9}$

17 Let N be the unique positive integer such that dividing 273436 by N leaves a remainder of 16 and dividing 272760 by N leaves a remainder of 15. What is the tens digit of N?

A. 0 B. 1 C. 2 D. 3 E. 4

Answer E

Solution Since N is unique, we know that $N = \gcd(273436 - 16, 272760 - 15) = \gcd(273420, 272745)$

Therefore, $N \mid (273420 - 272745) = 675 = 5^2 \times 3^3$.

Since $N \mid \gcd(272745, 675)$ and noting $25 \nmid 272745, 27 \nmid 272745$, we deduce $N \mid 5 \times 3^2 = 45$.

As $45 \mid 272745, 675$, we have $N = 45$.

18 The harmonic mean of a collection of numbers is the reciprocal of the arithmetic mean of the reciprocals of the numbers in the collection. For example, the harmonic mean of 4, 4, and 5 is

$$\frac{1}{\frac{1}{3}\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{5}\right)} = \frac{30}{7}.$$

What is the harmonic mean of all the real roots of the 4050th degree polynomial

$$\prod_{k=1}^{2025}(kx^2 - 4x - 3) = (x^2 - 4x - 3)(2x^2 - 4x - 3)(3x^2 - 4x - 3)\cdots(2025x^2 - 4x - 3)?$$

A. $-\frac{5}{3}$

B. $-\frac{3}{2}$

C. $-\frac{6}{5}$

D. $-\frac{5}{6}$

E. $-\frac{2}{3}$

Answer B

Solution Let $x_k^{(1)}$ and $x_k^{(2)}$ denote the roots of $kx^2 - 4x - 3$.

By Vieta's formulas: $\frac{1}{x_k^{(1)}} + \frac{1}{x_k^{(2)}} = \frac{x_k^{(1)} + x_k^{(2)}}{x_k^{(1)} \cdot x_k^{(2)}} = \frac{\frac{4}{k}}{-\frac{3}{k}} = -\frac{4}{3}$

Summing over all values: $\sum_{k=1}^{2025}\left(\frac{1}{x_k^{(1)}} + \frac{1}{x_k^{(2)}}\right) = -\frac{4}{3} \times 2025$

Computing the required expression: $\frac{4050}{\sum_{k=1}^{2025}\left(\frac{1}{x_k^{(1)}} + \frac{1}{x_k^{(2)}}\right)} = \frac{4050}{-\frac{4}{3} \times 2025} = -\frac{3}{2}.$

19 An array of numbers is constructed beginning with the numbers $-1\ 3\ 1$ in the top row. Each adjacent pair of numbers is summed to produce a number in the next row. Each row will begin and end with the numbers -1 and 1, respectively. The first three rows are shown below.

$$\begin{array}{ccccccccc} & & -1 & & 3 & & 1 & & \\ & -1 & & 2 & & 4 & & 1 & \\ -1 & & 1 & & 6 & & 5 & & 1 \end{array}$$

If the process continues, one of the rows will sum to $12,288$. In that row, what is the third number from the left?

A. -29 B. -21 C. -14 D. -8 E. -3

Answer A

Solution Let S_n represent the sum of row n. Since all the numbers in row $n+1$ can be seen as the sum of the two numbers above it (one number above for the leftmost -1 and rightmost 1), we have the recurrence relation $S_{n+1}=2S_n$, yielding $S_n=3\times 2^{n-1}$. As $12288=3\times 2^{12}$, it is the sum of the 13-th row.

Define a_n as the second element of row n, and b_n as the third element.

From the pattern: $a_{n+1}=a_n-1$ with $a_1=3$, which gives $a_n=4-n$ for $n\geq 1$.

With $b_n=b_{n-1}+a_{n-1}$ and $b_1=1$:

$$b_n-b_1=\sum_{k=2}^{n}a_{k-1}=\sum_{k=1}^{n-1}a_k=\sum_{k=1}^{n-1}(4-k)=4(n-1)-\frac{n(n-1)}{2}$$

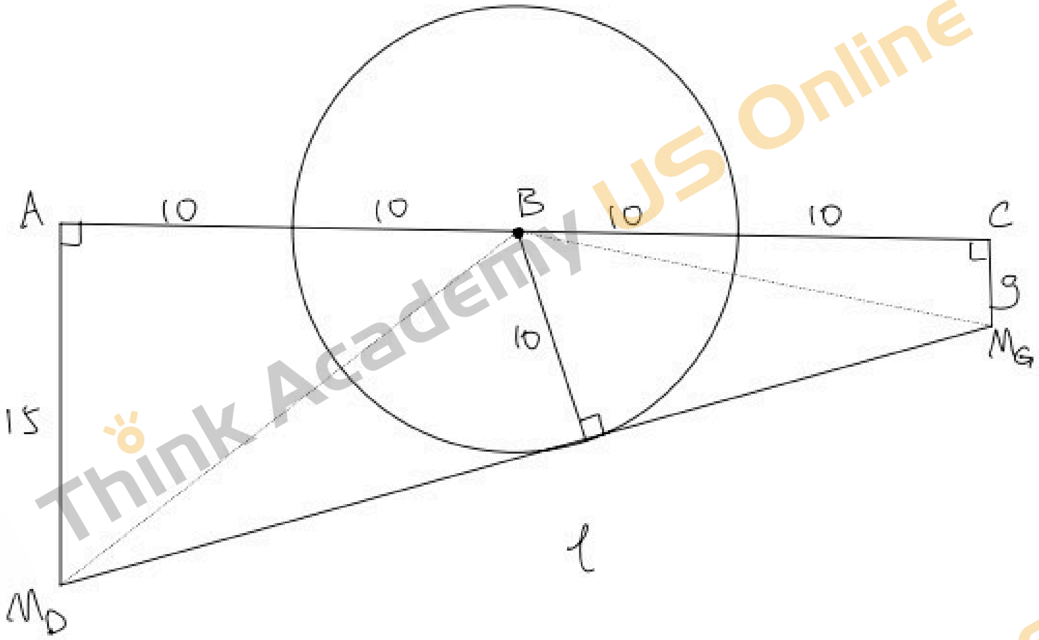
Therefore, $b_n=4n-3-\dfrac{n(n-1)}{2}$. $b_{13}=4\times 13-3-\dfrac{13\times 12}{2}=-29$

20 A silo (right circular cylinder) with diameter 20 meters stands in a field. MacDonald is located 20 meters west and 15 meters south of the center of the silo. McGregor is located 20 meters east and $g>0$ meters south of the center of the silo. The line of sight between MacDonald and McGregor is tangent to the silo. The value of g can be written as $\dfrac{a\sqrt{b}-c}{d}$, where a,b,c and d are positive integers, b is not divisible by the square of any prime, and d is relatively prime to the greatest common divisor of a and c. What is $a+b+c+d$?

A. 119 B. 120 C. 121 D. 122 E. 123

Answer A

Solution



Let ℓ denote the distance between points M_D and M_G. The trapezoid's area is:

$S_{\triangle BAM_D} + S_{\triangle BCM_G} + S_{\triangle BM_DM_G} = \frac{1}{2}(15 \times 20 + 20 \times 8 + 10 \times \ell)$

Also, trapezoid area = $\frac{1}{2}(15 + g) \times 40$.

Equating: $\frac{1}{2}(300 + 20g + 10\ell) = \frac{1}{2}(15 + g) \times 40$ gives $\ell = 2g + 30$.

Using the Pythagorean theorem: $(2g + 30)^2 = 40^2 + (15 - g)^2$.

Expanding: $4g^2 + 120g + 900 = 1600 + 225 - 30g + g^2$

This simplifies to $3g^2 + 150g - 925 = 0$.

Applying the quadratic formula: $\Delta = 150^2 + 4 \times 3 \times 925 = 33600$

$g = \dfrac{-150 \pm \sqrt{33600}}{6} = \dfrac{-75 \pm 20\sqrt{21}}{3}$

Since $g > 0$: $g = \dfrac{-75 + 20\sqrt{21}}{3}$.

21 A set of numbers is called $sum-free$ if whenever x and y are (not necessarily distinct) elements of the set, $x + y$ is not an element of the set. For example, $\{1, 4, 6\}$ and the empty set are sum-free, but $\{2, 4, 5\}$ is not. What is the greatest possible number of elements in a sum-free subset of $\{1, 2, 3, \cdots, 20\}$?

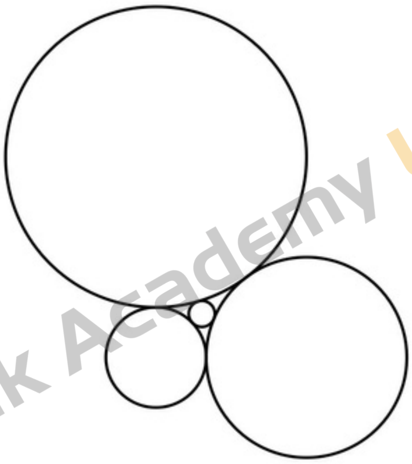
A. 8 B. 9 C. 10 D. 11 E. 12

Answer C

Solution The set $\{11, 12, \ldots, 20\}$ forms a sum-free subset with 10 elements.

Suppose there exists a sum-free subset $\{a_1, \ldots, a_{11}\}$ of 11 elements, with a_{11} the largest element. Then the differences $a_{11} - a_i$ for $1 \leq i \leq 10$ are all distinct from $\{a_1, \ldots, a_{11}\}$. Moreover, these differences are mutually distinct. The union of the 10 differences and the 11 original elements would form a 21-element subset of $\{1, 2, \ldots, 20\}$, which is a contradiction.

22 A circle of radius r is surrounded by three circles, whose radi are 1, 2, and 3, all externally tangent to the inner circle and externally tangent to each other, as shown in the diagram below.

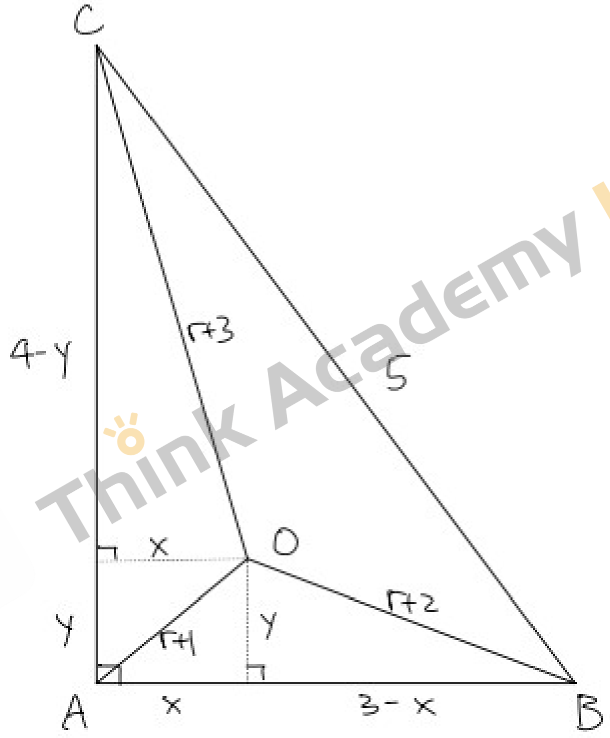


What is r?

A. $\frac{1}{4}$ B. $\frac{6}{23}$ C. $\frac{3}{11}$ D. $\frac{5}{17}$ E. $\frac{3}{10}$

Answer B

Solution



Let A, B, C denote the centers of the three circles, with O as the small circle's center.

$AC = 1 + 3 = 4, \quad AB = 1 + 2 = 3, \quad BC = 2 + 3 = 5$

Since $AC^2 + AB^2 = BC^2$, we have $\angle BAC = 90^\circ$.

Establish coordinates: $A(0,0), B(3,0), C(0,4)$.

Let $O = (x, y)$. Since $AO = 1 + r, BO = 2 + r, CO = 3 + r$: - Equation (1): $x^2 + y^2 = (1 + r)^2$ - Equation (2): $(x - 3)^2 + y^2 = (2 + r)^2$ - Equation (3): $x^2 + (y - 4)^2 = (3 + r)^2$

From equations (1) and (2): $x = 1 - \frac{r}{3}$.

From equations (1) and (3): $y = 1 - \frac{r}{2}$.

Substituting into (1): $\left(1 - \frac{r}{3}\right)^2 + \left(1 - \frac{r}{2}\right)^2 = (1 + r)^2$

This simplifies to $23r^2 + 132r - 36 = 0$, giving $r = \frac{6}{23}$ or $r = -6$.

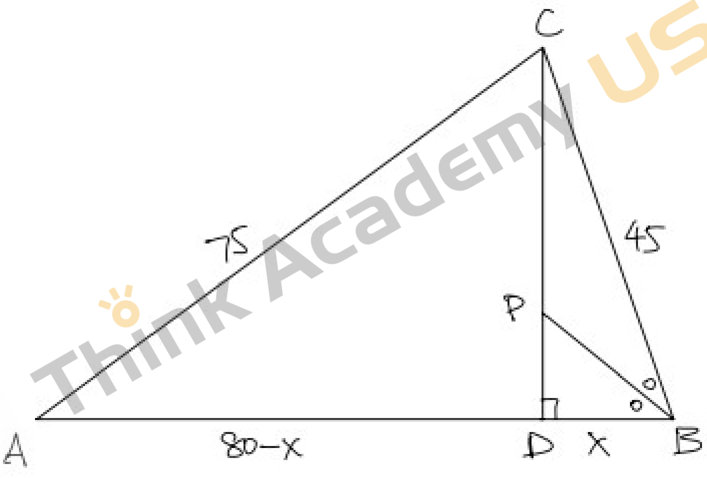
Since $r > 0$: $r = \frac{6}{23}$.

23 Triangle ΔABC has side lengths $AB = 80$, $BC = 45$, and $AC = 75$. The bisector of $\angle B$ and the altitude to side $\overline{AB}$ intersect at point P. What is BP?

A. 18 B. 19 C. 20 D. 21 E. 22

Answer D

Solution



Set $BD = x$, making $AD = 80 - x$. $BC^2 - BD^2 = CD^2 = AC^2 - AD^2$

This gives $45^2 - x^2 = 75^2 - (80 - x)^2$.

Solving: $x = \frac{35}{2}$, and $CD^2 = 45^2 - \left(\frac{35}{2}\right)^2 = \frac{25 \times 275}{4}$, so $CD = \frac{25\sqrt{11}}{2}$.

By the angle bisector theorem, $\frac{DP}{PC} = \frac{BD}{BC} = \frac{7}{18}$

So $DP = \frac{7}{18+7} \times CD = \frac{7\sqrt{11}}{2}$

Therefore: $BP^2 = BD^2 + DP^2 = 441$

Thus $BP = 21$.

24 Call a positive integer $fair$ if no digit is used more than once, it has no 0s, and no digit ins adjacent to two greater digits. For example, 196, 23, and 12463 are fair, but 1546, 320, and 34321 are not fair. How many fair positive integers are there?

A. 511 B. 2584 C. 9841 D. 17711 E. 19682

Answer C

Solution For k selected digits, let a_k count arrangements where no digit is adjacent to two larger digits.

Base cases: $a_1 = 1, a_2 = 2$.

For $k \geq 3$, the smallest digit must occupy either the first or last position.

Each placement corresponds bijectively to arranging $k-1$ digits with the same constraint, giving $a_k = 2a_{k-1}$, hence $a_k = 2^{k-1}$.

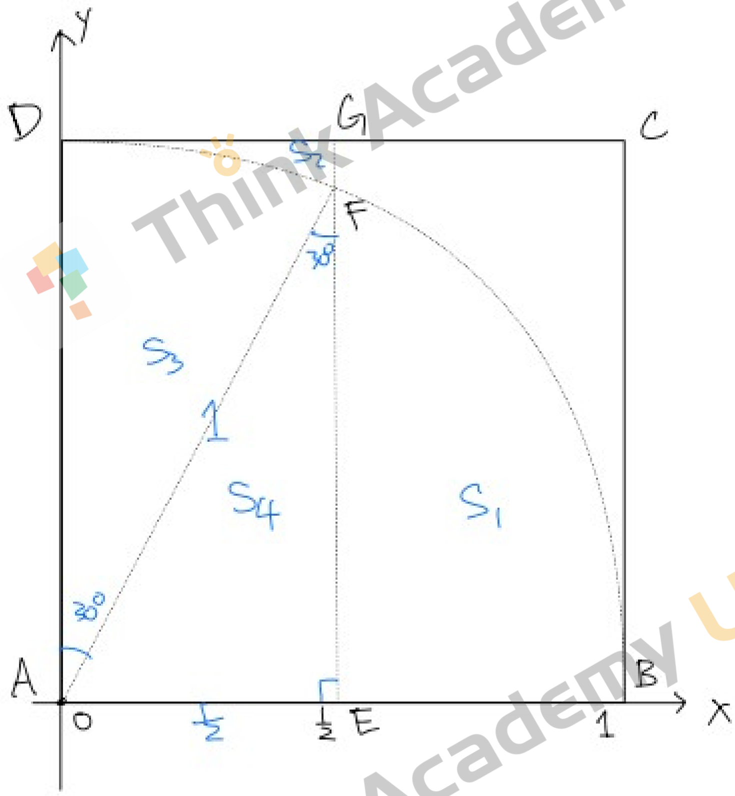
Total fair integers: $\sum_{k=1}^{9} \binom{9}{k} \cdot 2^{k-1} = \frac{\sum_{k=0}^{9} \binom{9}{k} \cdot 2^k - 1}{2} = \frac{(2+1)^9 - 1}{2} = \frac{3^9 - 1}{2} = 9841$

25 A point P is chosen at random inside square $ABCD$. The probability that $\overline{AP}$ is neither the shortest nor the longest side of $\triangle APB$ can be written as $\frac{a + b\pi - c\sqrt{d}}{e}$, where a, b, c, d, and e are positive integers, $gcd(a, b, c, e) = 1$, and d is not divisible by the square of any prime. What is $a + b + c + d + e$?

A. 25 B. 26 C. 27 D. 28 E. 29

Answer A

Solution



Establish coordinates: $A(0,0), B(1,0), C(1,1), D(0,1)$.

Scenario 1: $AB > AP > BP$. For point $P(x, y)$:

Condition $AP > BP$ implies $x > \frac{1}{2}$, and condition $AB > AP$ implies $AP < 1$.

Point P lies within the unit circle centered at A and to the right of $x = \frac{1}{2}$ which is EF. This defines region S_1.

Scenario 2: $AB < AP < BP$.

Condition $AP < BP$ implies $x < \frac{1}{2}$, and condition $AB < AP$ implies $AP > 1$.

Point P lies outside the unit circle and to the left of $x = \frac{1}{2}$. This defines region S_2.

Mark S_3 and S_4 as in the diagram too. Notice that $\angle OEF = 90^\circ$ and $\frac{OE}{OF} = \frac{1}{2}$. So $\angle OFE = \angle DAF = 30^\circ$

Thus $S_3 = \frac{30^\circ}{360^\circ}\pi \cdot 1^2 = \frac{\pi}{12}$, and $S_4 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$.

Also $S_2 + S_3 + S_4 = A_{DGEA} = \frac{1}{2}$, and $S_1 + S_3 + S_4 = \frac{90^\circ}{360^\circ}\pi \cdot 1^2 = \frac{\pi}{4}$.

Final result: $S_1 + S_2 = \frac{6 + \pi - 3\sqrt{3}}{12}$.