2025 AMC 10B

1 The instructions on a 350-gram bag of coffee beans say that proper brewing of a large mug of pour-over coffee requires 20 grams of coffee beans. What is the greatest number of properly brewed large mugs of coffee that can be made from the coffee beans in that bag?

A. 16 B. 17 C. 18 D. 19 E. 20

Answer B

Solution $\left\lfloor \frac{350}{20} \right\rfloor = \lfloor 17.5 \rfloor = 17$

2 Jerry wrote down the ones digit of each of the first 2025 positive squares: $1, 4, 9, 6, 5, 6, \cdots$. What is the sum of all the numbers Jerry wrote down?

A. 9025 B. 9070 C. 9090 D. 9115 E. 9160

Answer D

Solution $(10a+b)^2 = 100a^2 + 20ab + b^2 \equiv b^2 \pmod{10}, \quad a, b \in \mathbb{Z}$

Let $f(n)$ be the ones digit of n, then $f(10a+b) = f(b)$.

$\sum_{n=1}^{2025} f(n) = 202 \cdot \sum_{n=1}^{10} f(n) + \sum_{n=1}^{5} f(n)$ which is

$202 \times (1+4+9+6+5+6+9+4+1+0) + (1+4+9+6+5) = 202 \times 45 + 25 = 9115.$

3 A Pascal-like triangle has 10 as the top row and 10 followed by 1 as the second row. In each subsequent row the first number is 10, the last number is 1, and, as in the standard Pascal Triangle, each other number in the row is the sum of the two numbers directly above it. The first four rows are shown below.

$$\begin{array}{ccccccc} & & & 10 & & & \\ & & 10 & & 1 & & \\ & 10 & & 11 & & 1 & \\ 10 & & 21 & & 12 & & 1 \end{array}$$

What is the sum of the digits of the sum of the numbers in the 11th row?

A. 11 B. 13 C. 14 D. 16 E. 17

Answer D

Solution Let S_k be the sum of k-th row.

Then $S_k = 2 \cdot S_{k-1} - (10+1) + (10+1) = 2S_{k-1}$ for $k \geq 3$.

Since $S_2 = 11$, we have $S_k = 11 \cdot 2^{k-2}$ for $k \geq 2$.

$S_{11} = 11 \times 2^9 = 11 \times 512 = 5632$

4 The value of the two-digit number $\underline{a}\,\underline{b}$ in base seven equals the value of the two-digit number $\underline{b}$ $\underline{a}$ in base nine. What is $a+b$?

A. 7 B. 9 C. 10 D. 11 E. 14

Answer A

Solution $(ab)_7 = (ba)_9$ implies $7a + b = 9b + a$ where $0 \leq a, b < 7$

This gives $6a = 8b$, then $3a = 4b$, so $a = 4k$ and $b = 3k$ for $k \in \mathbb{Z}$.

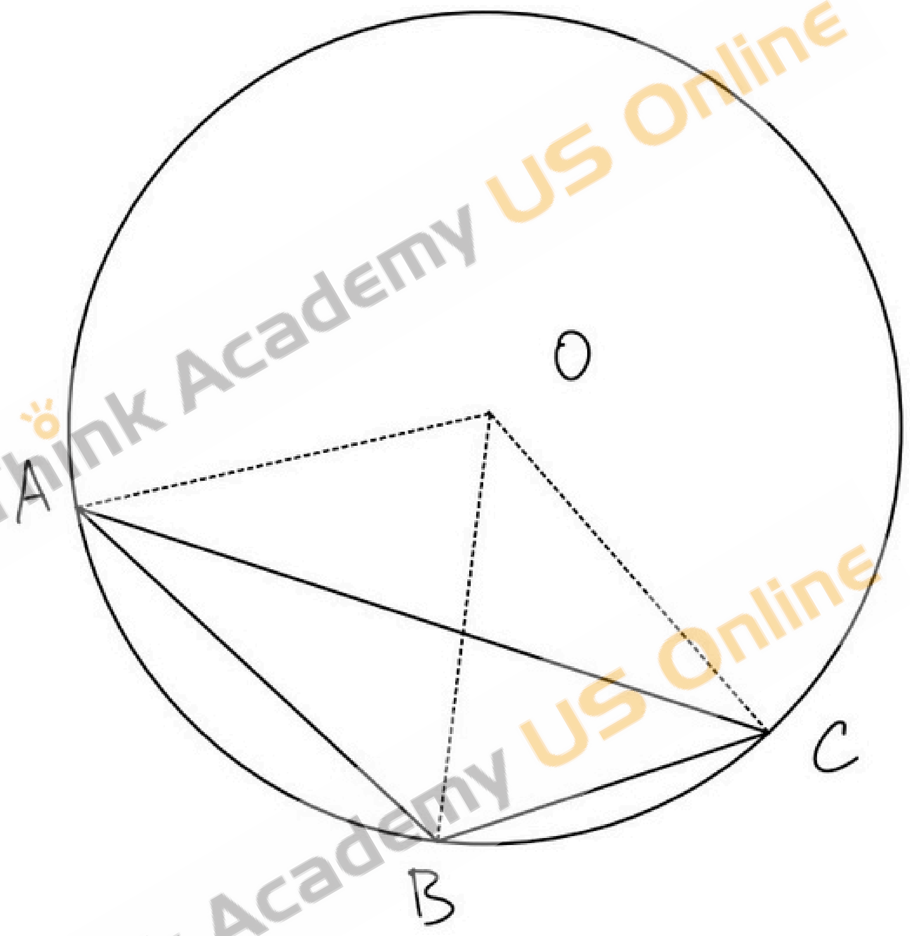
Since $a \in (0, 7)$, we have $k = 1$, thus $a = 4$ and $b = 3$.

5 In $\triangle ABC$, $AB = 10$, $AC = 18$, and $\angle B = 130°$. Let O be the center of the circle containing points A, B, and C. What is the degree measure of $\angle CAO$?

A. 20 B. 30 C. 40 D. 50 E. 60

Answer C

Solution



$\angle ABC = 130^\circ \implies \angle AOC = 2 \times (180^\circ - \angle ABC) = 2 \times 50^\circ = 100^\circ$

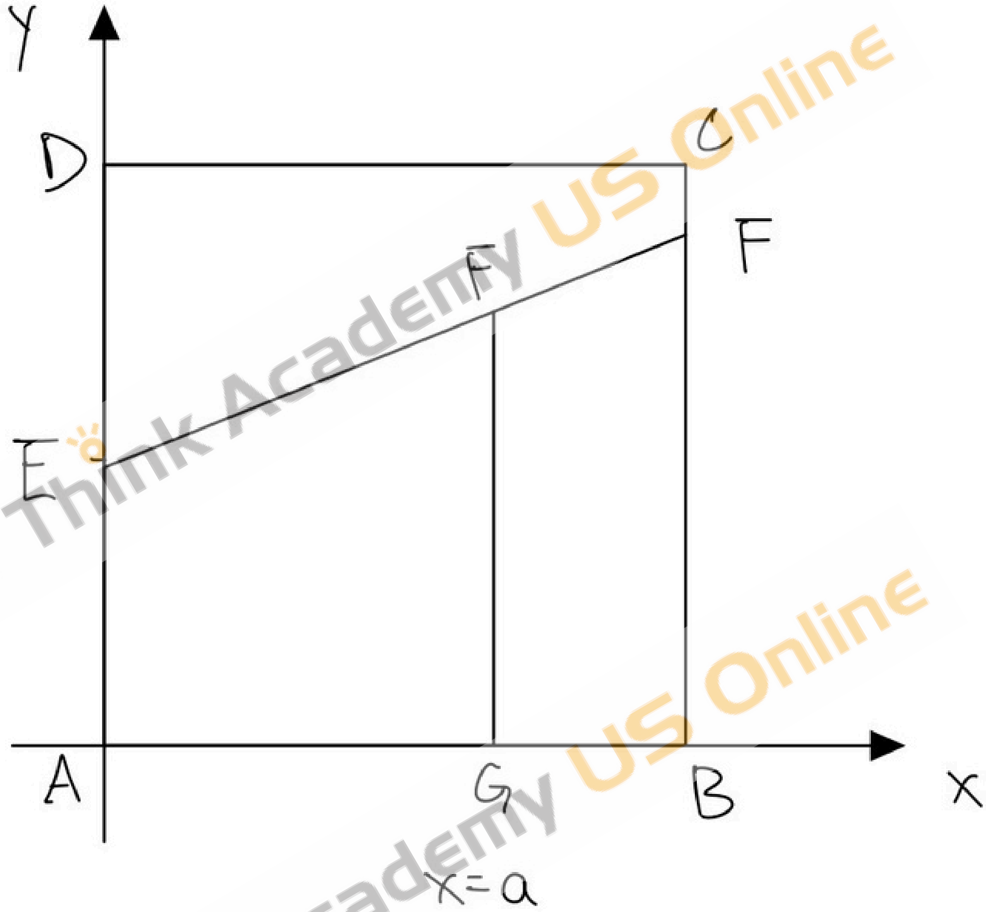
$OA = OC \implies \angle OAC = \dfrac{1}{2} \times (180^\circ - \angle AOC) = \dfrac{1}{2} \times (180^\circ - 100^\circ) = 40^\circ$

6 The line $y = \dfrac{1}{3}x + 1$ divides the square region defined by $0 \leqslant x \leqslant 2$ and $0 \leqslant y \leqslant 2$ into an upper region and a lower region. The line $x = a$ divides the lower region into two regions of equal area. Then a can be written as $\sqrt{s} - t$, where s and t are positive integers. What is $s + t$?

A. 18 B. 19 C. 20 D. 21 E. 22

Answer C

Solution



Points: $E(0,1)$, $F(2,\frac{5}{3})$, $G(a,0)$, $H(a,\frac{1}{3}a+1)$, $A(0,0)$, $B(2,0)$.

Given $S_{AGHE}=\frac{1}{2}S_{ABFE}$:

$$\frac{1}{2}\cdot a\cdot\left(\frac{1}{3}a+1+1\right)=\frac{1}{2}\cdot\frac{1}{2}\cdot 2\times\left(1+\frac{5}{3}\right)$$

This gives $a^2+6a-8=0$, so $a=\pm\sqrt{17}-3$.

Since $a>0$, we have $a=\sqrt{17}-3$.

7 Frances stands 15 meters directly south of a locked gate in a fence that runs east-west. Immediately behind the fence is a box of chocolates, located x meters east of the locked gate. An unlocked gate lies 9 meters east of the box, and another unlocked gate lies 8 meters west of the locked gate. Frances can reach the box by walking toward an unlocked gate, passing through it, and walking toward the box. It happens that the total distance Frances would travel would be the same via either unlocked gate. What is value of x?

A. $3\frac{2}{7}$

B. $3\frac{3}{7}$

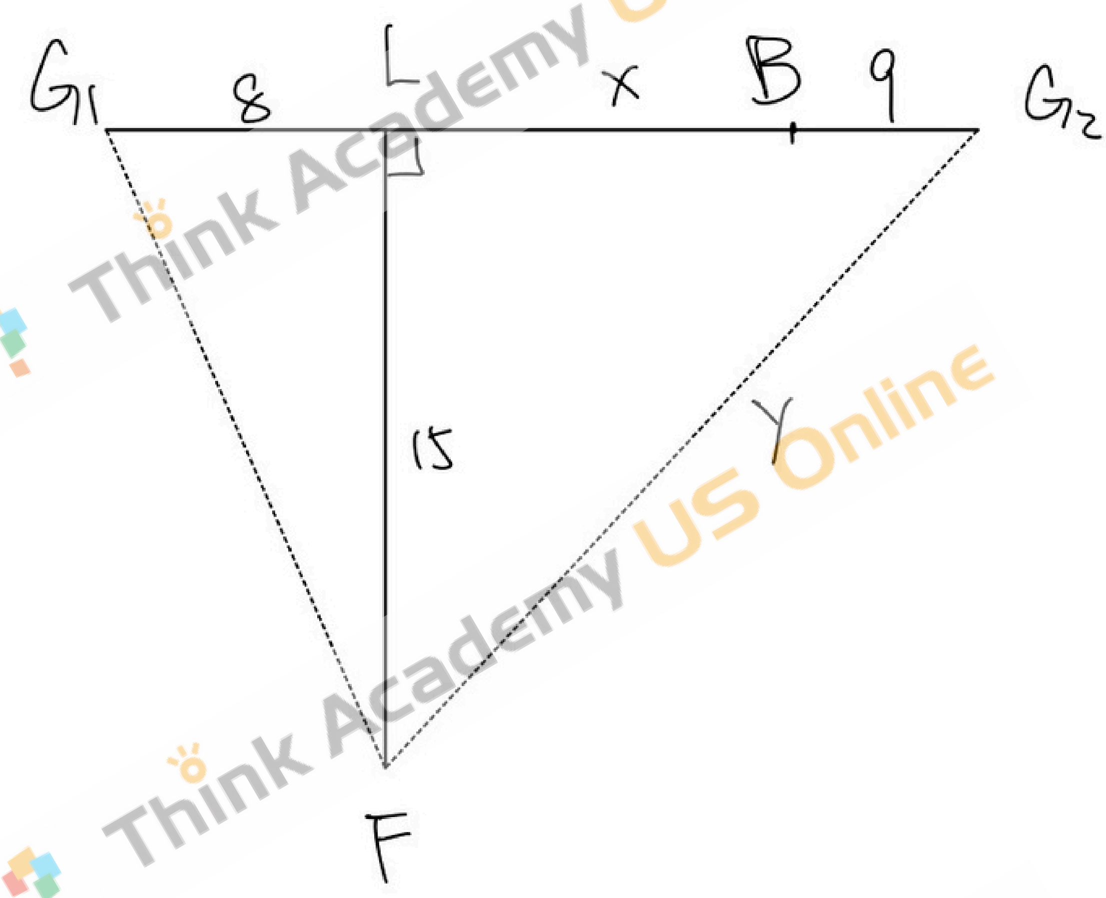
C. $3\frac{4}{7}$

D. $3\frac{5}{7}$

E. $3\frac{6}{7}$

Answer C

Solution



Let $G_2F = y$. Here, $FL \perp G_1G_2$, $BL = x$, $BG_2 = 9$, $LG_1 = 8$, $LF = 15$.

Then $G_1F = \sqrt{LF^2 + LG_1^2} = \sqrt{8^2 + 15^2} = 17$

$BG_1 + FG_1 = 8 + x + 17 = 25 + x$

Also, $9 + y = 25 + x$, which gives $y = 16 + x$.

Since $y^2 = (x+9)^2 + 15^2$, we have $(16+x)^2 = (x+9)^2 + 15^2$, which gives $x = \frac{25}{7}$.

8 Emmy says to Max, “I ordered 36 math club sweatshirts today." Max asks, “How much did each shirt cost?" Emmy responds, “I'll give you a hint. The total cost was \$ $\underline{A}\,\underline{B}\,\underline{B}\;\underline{B}\,\underline{A}$, where A and B are digits and $A \neq 0$." After a pause, Max says, “That was a good price." What is $A + B$?

A. 7 B. 8 C. 11 D. 14 E. 15

Answer C

Solution Given $36 \mid (ABBBA)_{10}$, we need $9 \mid 3B+2A$ and $4 \mid (BA)_{10}$

From $3 \mid 3B+2A$, we get $3 \mid A$. Since $(BA)_{10}$ is even, A is even, so $2 \mid A$. Thus, $A=6$.

From $9 \mid 3B+2A=3B+12$, we get $3 \mid B+4$, so $B \in \{2,5,8\}$.

From $4 \mid 10B+A=10B+6$, we get $2 \mid 5B+3$, which means B is odd.

Therefore $B=5$.

9 How many ordered triples of integers (x,y,z) satisfy the following system of inequalities?

$$-x-y-z \leqslant -2$$

$$-x+y+z \leqslant 2$$

$$x-y+z \leqslant 2$$

$$x+y-z \leqslant 2$$

A. 4 B. 8 C. 11 D. 15 E. 17

Answer C

Solution System of inequalities:

$x+y+z \geq 2$

$-x+y+z \leq 2$

$x-y+z \leq 2$

$x+y-z \leq 2$

Combining the first and second inequalities, we get $x \geq 0$. By symmetry, $x,y,z \geq 0$.

Suppose $0 \leq x \leq y \leq z$ (WLOG).

From the second inequality, $y+z \leq 2+x \leq 2+y$, which gives $z \leq 2$.

Thus, $0 \leq x \leq y \leq z \leq 2$.

Case 1: $x=0$, then from the first and second inequalities, $y+z=2$. Solutions:

$(y,z)=(0,2)$ or $(1,1)$.

Case 2: $x=1$, then from the second inequality, $y+z \leq 3$. Since $1 \leq y \leq z \leq 2$, solutions:

$(y,z)=(1,1)$ or $(1,2)$.

Case 3: $x=2$, then $(x,y,z)=(2,2,2)$.

In total:

$(1,1,1)$ and $(2,2,2)$ each give 1 solution

$(0,0,2)$, $(0,1,1)$, $(1,1,2)$ each give 3 solutions

Total number = $1\times 2+3\times 3=11$

10 Let $f(n)=n^3-5n^2+2n+8$,and let $g(n)=n^3-6n^2+5n+12$. What is the sum of all integer values of n for which $\frac{f(n)}{g(n)}$ is also an integer?

A. 2 B. 3 C. 4 D. 5 E. 6

Answer A

Solution We have the following observations: $\frac{f(n)}{g(n)}-1=\frac{f(n)-g(n)}{g(n)}\in\mathbb{Z}$

$f(n)-g(n)=n^2-3n-4=(n-4)(n+1)$

$g(n)=(n+1)(n-4)(n-3)$

$\frac{f(n)-g(n)}{g(n)}=\frac{1}{n-3}$ when $n\neq -1,4,3$

For $\frac{1}{n-3}\in\mathbb{Z}$, we need $n-3=\pm 1$, so $n=2$ or 4.

However, $n\neq 4$, thus $n=2$.

11 On Monday, 6 students went to the tutoring center at the same time, and each one was randomly assigned to one of the 6 tutors on duty. On Tuesday, the same 6 students showed up, the same 6 tutors were on duty, and the students were again randomly assigned to the tutors. What is the probability that exactly 2 students met with the same tutor both Monday and Tuesday?

A. $\frac{1}{16}$ B. $\frac{3}{16}$ C. $\frac{1}{4}$ D. $\frac{3}{8}$ E. $\frac{1}{2}$

Answer B

Solution For the general case with n students and n tutors:

Let S_i be the case where the i-th student has the same tutor, then $|S_i| = (n-1)!$ and

$|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}| = (n-k)!$.

$|(S_1 \cup \cdots \cup S_n)^c| = n! - |S_1 \cup \cdots \cup S_n|$

$= n! - \left(\sum_{i=1}^{n} |S_i| - \sum_{1 \le i_1 < i_2 \le n} |S_{i_1} \cap S_{i_2}| + \cdots + (-1)^{n+1} |S_1 \cap \cdots \cap S_n| \right)$

$= n! - \left(n \times (n-1)! - \binom{n}{2} \cdot (n-2)! + \cdots + (-1)^{n+1} \cdot 1 \right)$

The k-th term inside the bracket equals $(-1)^{k+1} \cdot \binom{n}{k} \cdot (n-k)! = (-1)^{k+1} \cdot \frac{n!}{k!}$.

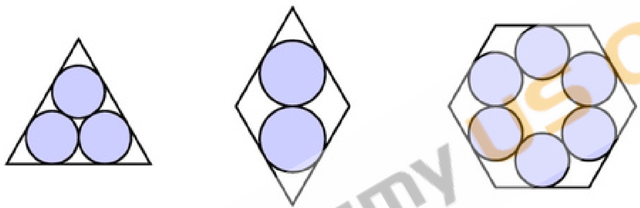
This gives $n! - \sum_{k=1}^{n} (-1)^{k+1} \cdot \frac{n!}{k!} = n! \cdot \left(\sum_{k=0}^{n} (-1)^k \cdot \frac{1}{k!} \right)$

For any 2 students, the total ways of assigning so that they get the same tutor, are the total ways of assigning the remaining 4 tutors so no students among these 4 get the same tutor:

$4! \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right) = 9$

Therefore, the probability $= \frac{\binom{6}{2} \times 9}{6!} = \frac{3}{16}$

12 The figure below shows an equilateral triangle, a rhombus with a 60° angle, and a regular hexagon, each of them containing some mutually tangent congruent disks. Let T, R, and H, respectively, denote the ratio in each case of the total area of the disks to the area of the enclosing polygon.

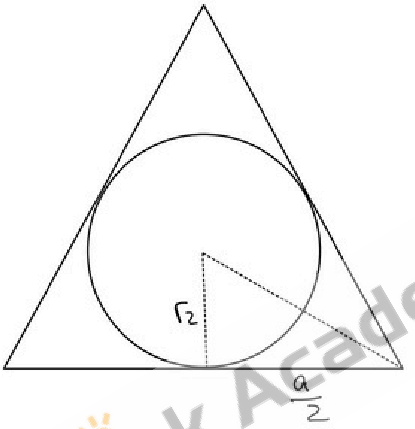


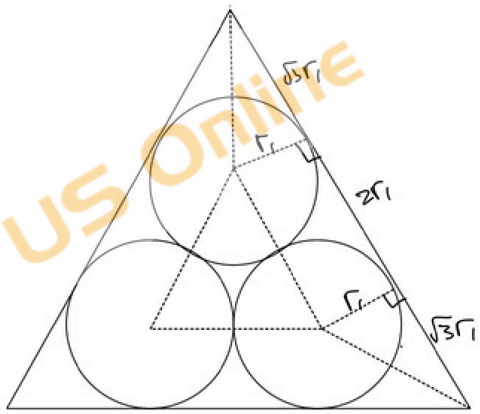
Which of the following is true?

A. $T = R = H$ B. $H < R = T$ C. $H = R < T$ D. $H < R < T$ E. $H < T < R$

Answer C

Solution





Let a be the length of a side of an equilateral triangle, r_1 be the radius in the 3-tangent-circle case, and r_2 be the one-tangent-circle case.

It can be checked that both R and H equal the ratio of the r_2 case.

Then: $a = 2\sqrt{3}r_1 + 2r_1 = 2(\sqrt{3}+1)r_1$ $a = 2\sqrt{3}r_2$

$$\frac{T}{R} = \frac{3\pi r_1^2}{\pi r_2^2} = \frac{3\times 3}{(\sqrt{3}+1)^2} = \frac{9}{4+2\sqrt{3}} > 1$$

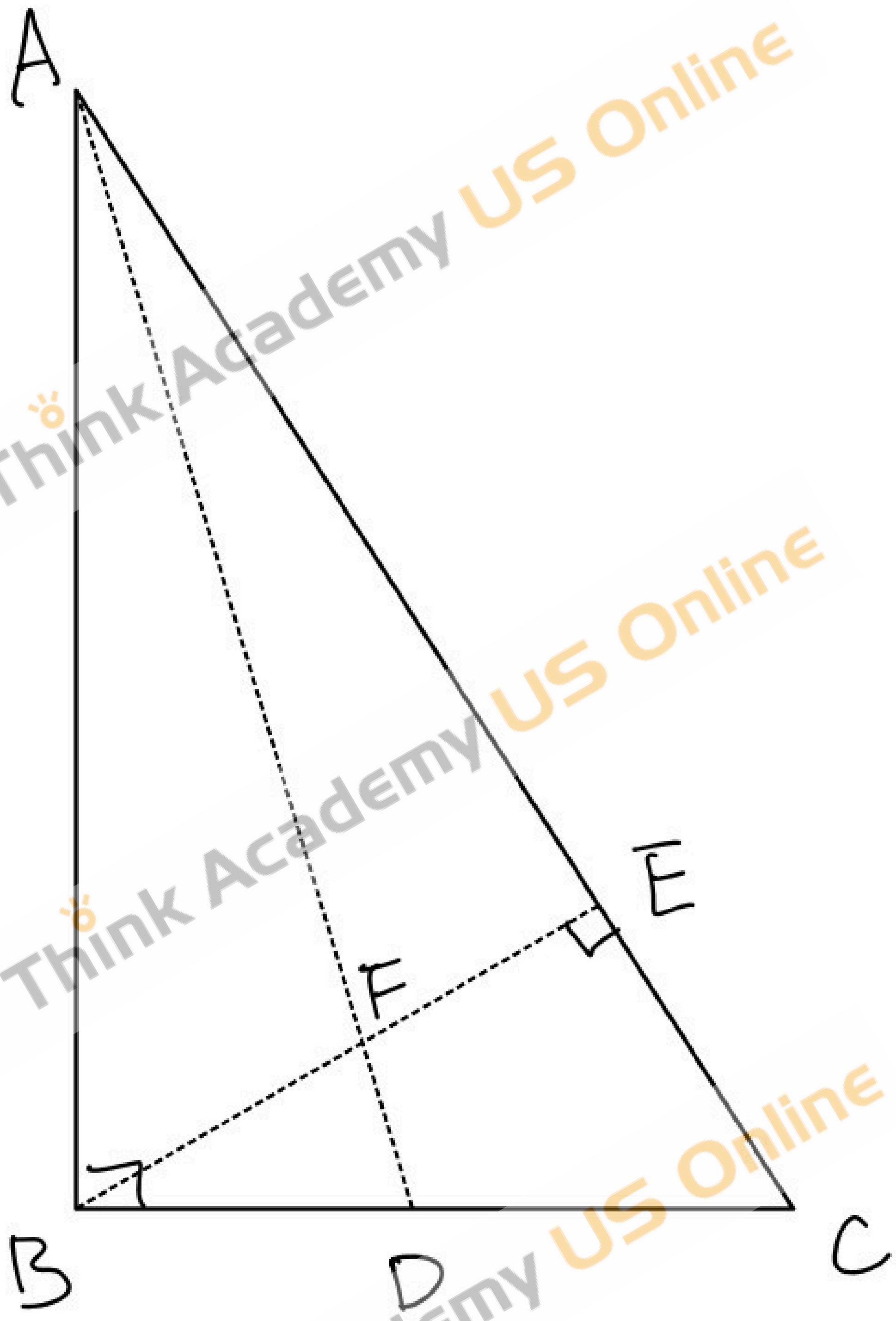
Thus $T > R = H$.

13 The altitude to the hypotenuse of a $30-60-90^{\circ}$ right triangle is divided into two segment of lengths $x < y$ by the median to the shortest side of the triangle. What is the ratio $\frac{x}{x+y}$?

A. $\frac{3}{7}$ B. $\frac{\sqrt{3}}{4}$ C. $\frac{4}{9}$ D. $\frac{5}{11}$ E. $\frac{4\sqrt{3}}{15}$

Answer A

Solution



ith Menelaus Theorem on $\triangle BCE$:

$$\frac{EF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CA}{AE} = 1$$

Since $BD = CD$:

$$\frac{AC}{AE} = \frac{AC}{AB} \cdot \frac{AB}{AE} = \frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}} = \frac{4}{3}$$

$\frac{EF}{FB} = \frac{AE}{AC} = \frac{3}{4}$ Thus $\frac{x}{x+y} = \frac{3}{3+4} = \frac{2}{7}$

14 Nine athletes, no two of whom are the same height, try out for the basketball team. One at a time, they draw a wristband at random, without replacement, from a bag containing 3 blue bands, 3 red bands, and 3 green bands. They are divided into a blue group, a red group, and a green group. The tallest member of each group is named the group captain. What is the probability that the group captains are the three tallest athletes?

A. $\frac{2}{9}$ B. $\frac{2}{7}$ C. $\frac{9}{28}$ D. $\frac{1}{3}$ E. $\frac{3}{8}$

Answer C

Solution Total number of assigning bands:

$$\binom{9}{3\ 3\ 3}=\frac{9!}{3!\cdot 3!\cdot 3!}$$

For the three tallest, the number of ways to assign different bands is $3!$, and for the remaining 6 players:

$$\binom{6}{2\ 2\ 2}=\frac{6!}{2!\cdot 2!\cdot 2!}$$

Thus, the probability:

$$=\frac{3!\cdot\frac{6!}{2!\cdot2!\cdot2!}}{\frac{9!}{3!\cdot3!\cdot3!}}=\frac{9}{28}$$

15 The sum $\sum_{k=1}^{\infty}\frac{1}{k^3+6k^2+8k}$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a+b$?

A. 89 B. 97 C. 102 D. 107 E. 129

Answer D

Solution $\sum_{k=1}^{\infty}\frac{k}{k(k+2)(k+4)}=\sum_{k=1}^{\infty}\frac{1}{(k+2)(k+4)}$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{(k+4)-(k+2)}{(k+2)(k+4)}=\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k+2}-\frac{1}{k+4}\right)$$

$$=\frac{1}{2}\cdot\left(\frac{1}{3}+\frac{1}{4}\right)$$

Similarly:

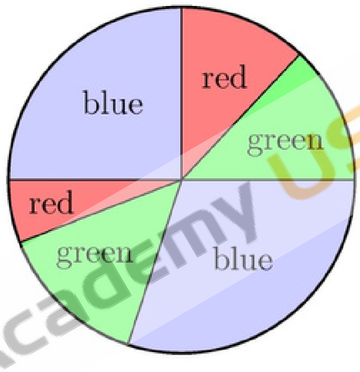
$$\sum_{k=1}^{\infty} \frac{k+4}{k(k+2)(k+4)} = \sum_{k=1}^{\infty} \frac{1}{k(k+2)} = \frac{1}{2} \cdot \left(1 + \frac{1}{2}\right)$$

Therefore:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+4)} = \frac{1}{4} \cdot \left(\sum_{k=1}^{\infty} \frac{k+4}{k(k+2)(k+4)} - \sum_{k=1}^{\infty} \frac{k}{k(k+2)(k+4)}\right)$$

$$= \frac{1}{8} \cdot \left(1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right) = \frac{11}{96}$$

16 A circle has been divided into 6 sectors of 6 different sizes. Then 2 of the sectors are painted red, 2 painted green, and 2 painted blue so that no two neighboring sectors are painted the same color. One such coloring is shown below.

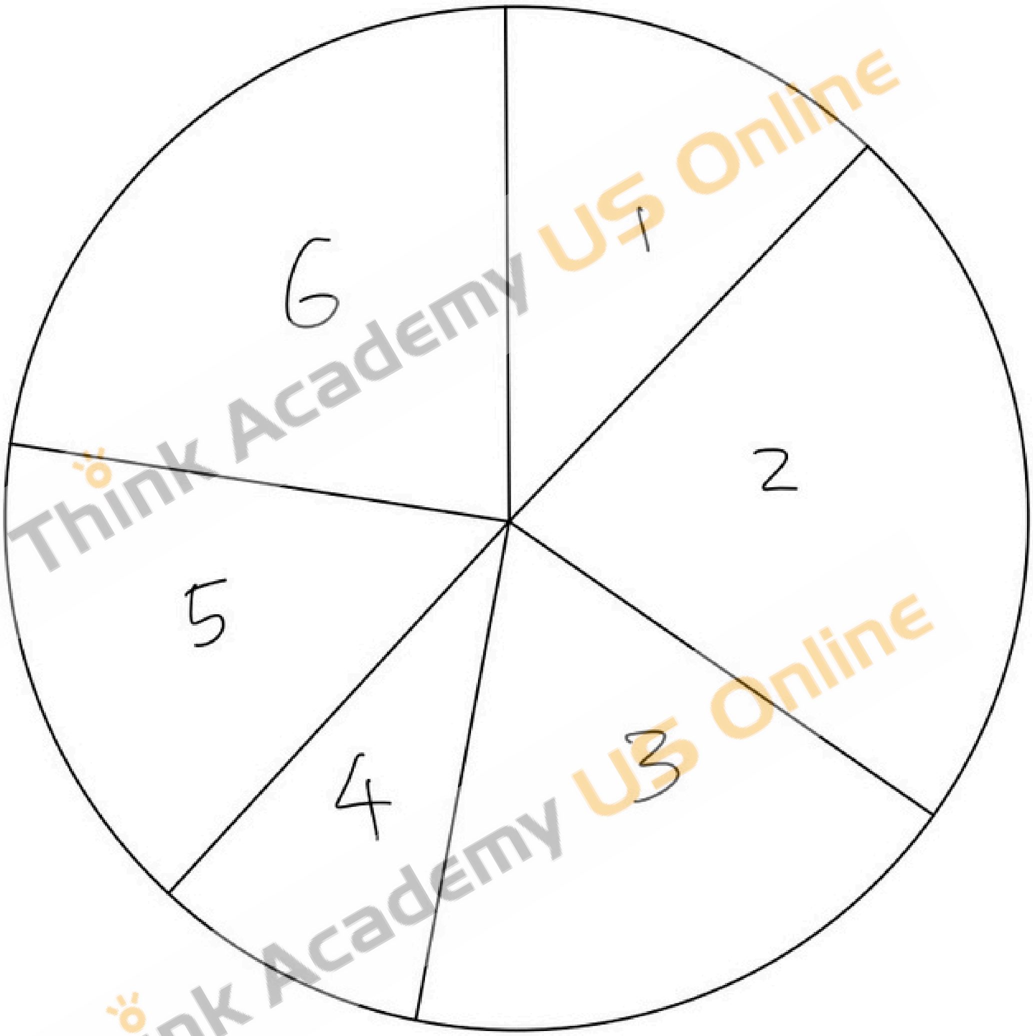


How many different colorings are possible?

A. 12 B. 16 C. 18 D. 24 E. 28

Answer D

Solution



3 ways to color region 1. When it is red, the other red can only be region 3, 4 or 5.

Case 1: Region 3 is red, then regions 4 and 6 have the same color, regions 2 and 5 have the same. 2 ways of coloring.

Case 2: Region 4 is red, then 2 ways to color regions 2 and 3 with different colors, and likewise for regions 5 and 6. There should be $2 \times 2 = 4$ ways.

Case 3: Region 5 is red, same as Case 1 with 2 ways.

Thus, the total ways $= 3 \times (2 + 4 + 2) = 24$.

17 Consider a decreasing sequence of n positive integers$s x_1 > x_2 > x_3 > \cdots > x_n$ that satisfies the following two conditions:

· The average (arithmetic mean) of the first 3 terms in the sequence is 2025

· For all $4 \leqslant k \leqslant n$ the average of the first k terms in the sequence is 1 less than the average of the first $k-1$ terms in the sequence.

What is the greatest possible value of n?

A. 1013 B. 1014 C. 1016 D. 2016 E. 2025

Answer B

Solution Let a_k be the average of the first k terms.

Given $a_3 = 2025$ and $a_k = a_{k-1} - 1$ for $k \geq 4$.

Thus $a_k = 2025 - (k-3) = 2028 - k$ for $k \geq 3$.

$x_k = k \cdot a_k - (k-1) \cdot a_{k-1} = a_k + (k-1) \cdot (a_k - a_{k-1})$

$= 2028 - k - k + 1 = 2029 - 2k$ for $k \geq 4$

For $x_k > 0$, we need $2029 - 2k > 0$, which gives $k \leq 1014$.

18 What is the ones digit of the sum $\lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \lfloor\sqrt{3}\rfloor + \cdots + \lfloor\sqrt{2024}\rfloor + \lfloor\sqrt{2025}\rfloor$?

(Recall that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

A. 1 B. 2 C. 3 D. 5 E. 8

Answer D

Solution $\sum_{n=1}^{2025} \lfloor\sqrt{n}\rfloor = \sum_{\substack{1 \leq n \leq 2025 \\ k^2 \leq n < (k+1)^2}} k$

For $1 \leq k \leq 44$, there are $(k+1)^2 - k^2 = 2k+1$ terms of value k.

For $k = 45$, there is exactly one term.

Thus:

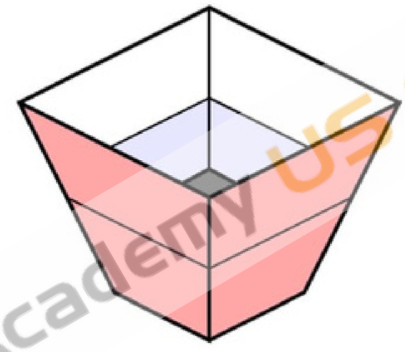
$$\sum_{k=1}^{44} k \cdot (2k+1) + 45 = 2\sum_{k=1}^{44} k^2 + \sum_{k=1}^{44} k + 45$$

$$= 2 \times \frac{1}{6} \times 44 \times 45 \times 89 + \frac{1}{2} \times 44 \times 45 + 45$$

Both of the first 2 terms have 0 as ones digit, so the answer is 5.

19 A container has a 1×1 square bottom, a 3×3 open square top, and four congruent trapezoidal sides, as shown. Starting when the container is empty, a hose that runs water at a constant rate

takes 35 minutes to fill the container up to the midline of the trapezoids.

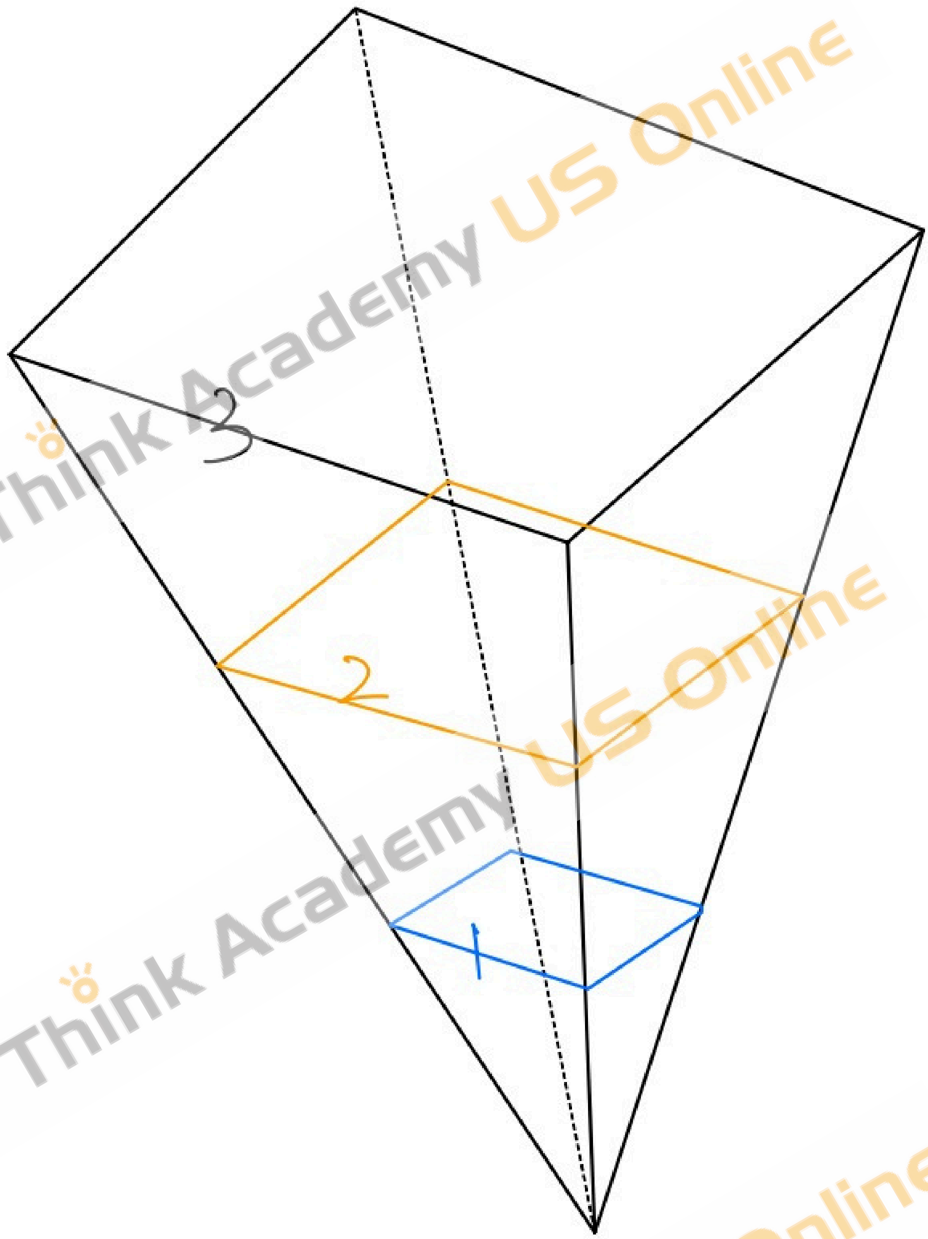


How many more minutes will it take to fill the remainder of the container?

A. 70 B. 85 C. 90 D. 95 E. 105

Answer D

Solution



Extend the lines so the sides become triangles.

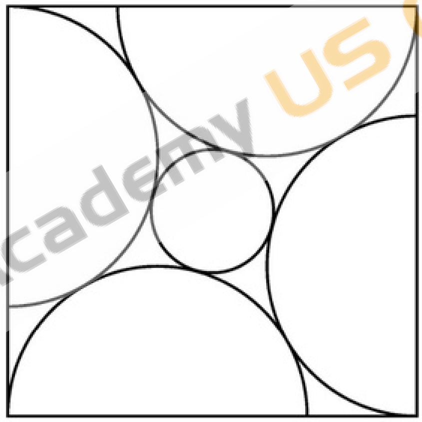
Let h be the height of half the container; then the extended cone also has height h since the side lengths of the squares are 1, 2, 3 respectively.

$$\frac{V_{\text{remained}}}{V_{\text{filled}}} = \frac{\frac{1}{3}\cdot 2h\cdot 2^2 - \frac{1}{3}\cdot 2h\cdot 1^2}{\frac{1}{3}\cdot 2h\cdot 1^2 - \frac{1}{3}\cdot h\cdot 1^2} = \frac{27-8}{8-1} = \frac{19}{7}$$

Therefore, the time $= 35 \times \dfrac{19}{7} = 95$ mins.

20 Four congruent semicircles are inscribed in a square of side length 1 so that their diameters are on the sides of the square, one endpoint of each diameter is at a vertex of the square, and

adjacent semicircles are tangent to each other. A small circle centered at the center of the square is tangent to each of the four semicircles, as shown below.

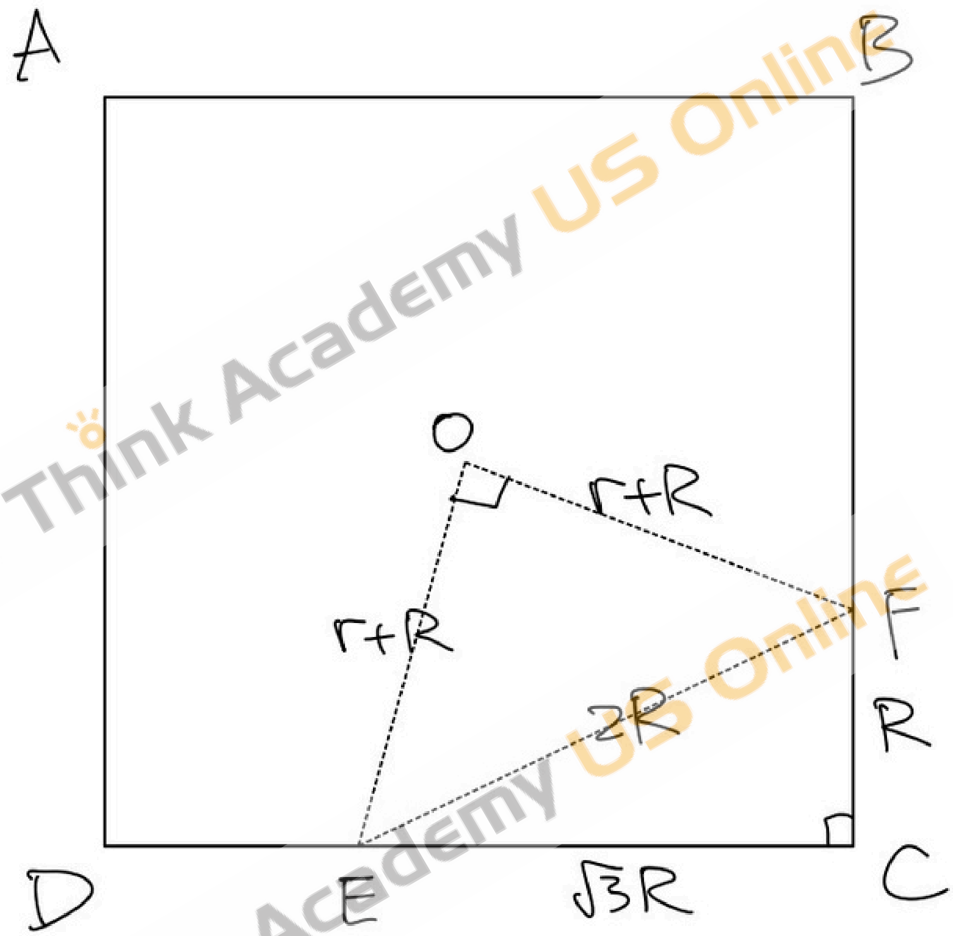


The diameter of the small circle can be written as $(\sqrt{a}+b)(\sqrt{c}+d)$, where a, b, c, and d are integers. What is $a+b+c+d$?

A. 3 B. 5 C. 8 D. 9 E. 11

Answer A

Solution



Let r be the radius of the small circle and R be the larger radius.

Let O, E, F be the centers, then $OE = OF = r + R$ and $EF = 2R$.

Since $CF = R$ and $\angle ECF = 90^\circ$, we have $\angle CFE = 60^\circ$ and $CE = \sqrt{3}R$.

Then:

$$R + \sqrt{3}R = 1 \implies R = \frac{1}{\sqrt{3}+1}$$

Since $\angle FOE = 90^\circ$ by symmetry, $EF = \sqrt{2}OE$, i.e., $2R = \sqrt{2}(R + r)$.

$$r = (\sqrt{2}-1)R = \frac{\sqrt{2}-1}{\sqrt{3}+1} = \frac{1}{2}(\sqrt{2}-1)(\sqrt{3}-1)$$

$$\implies 2r = (\sqrt{2}-1)(\sqrt{3}-1)$$

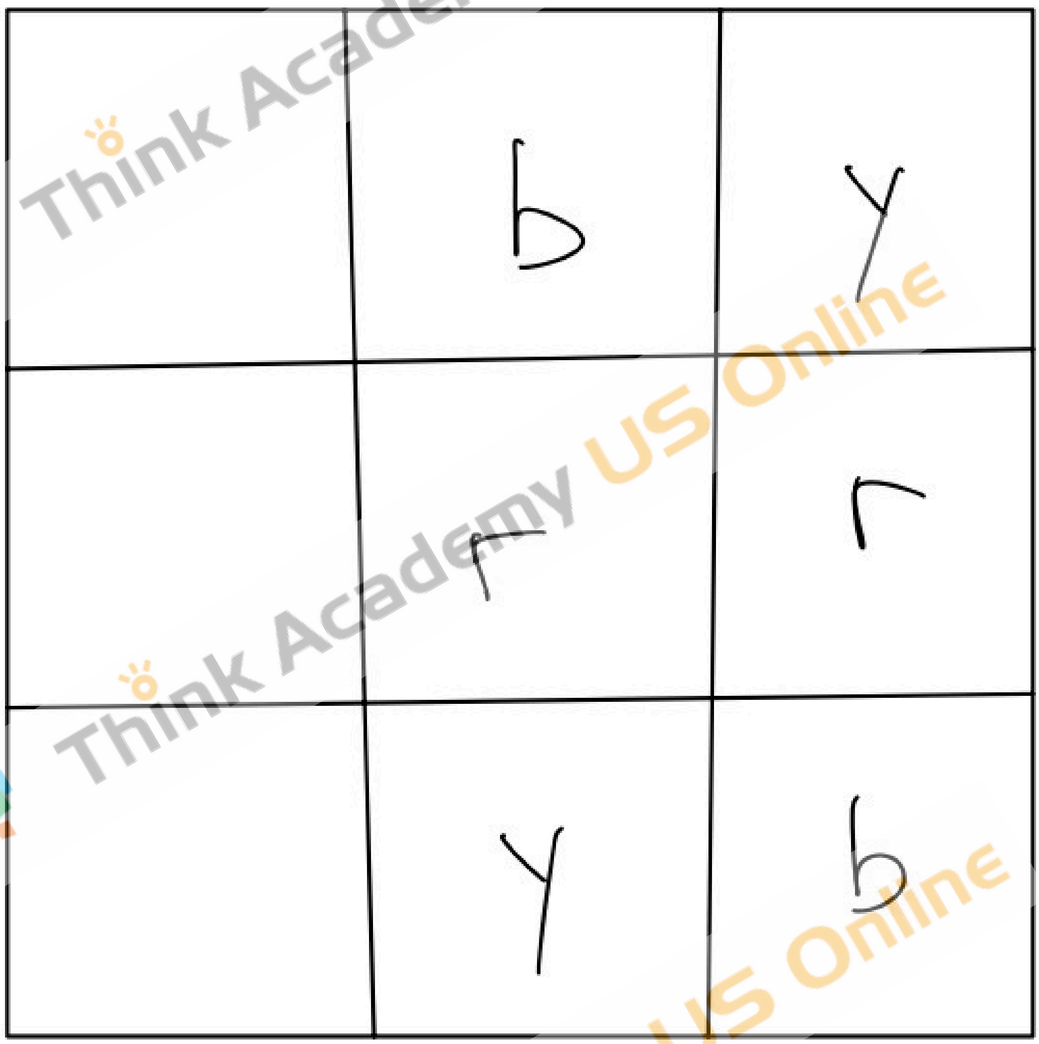
21 Each of the 9 squares in a 3×3 grid is to be colored red, blue, or yellow in such a way that each red square shares an edge with at least one blue square, each blue square shares an edge with at least one yellow square, and each yellow square shares an edge with at least one red square. Colorings that can be obtained from one another by rotations and/or reflections are

to be considered the same. How many different colorings are possible?

A. 3 B. 9 C. 12 D. 18 E. 27

Answer C

Solution



Assume the color of row i, column j is $c(i,j)$.

Consider the center grid when it is red.

At least one of $c(1,2)$, $c(3,2)$, $c(2,1)$, $c(2,3)$ is blue.

Suppose $c(1,2)$ = blue (WLOG).

At least one of $c(1,1)$, $c(1,3)$ is yellow, assume $c(1,3)$ is yellow (WLOG).

Then considering $c(1,3)$, at least one of $c(1,2)$ and $c(2,3)$ must be red, meaning $c(2,3)$ = red.

Considering $c(2,3)$, one of its neighbors is blue, meaning $c(3,3)$ = blue.

Considering $c(3,3)$, one of its neighbors is yellow, meaning $c(3,2)$ = yellow.

For the remaining 3 grids, there are 3 cases:

Case 1: $c(2,1)$ = red, then $c(1,1)$ = blue or $c(3,1)$ = blue.

If $c(1,1)$ = blue, its neighbor does not have yellow (contradiction!).

Thus $c(3,1)$ = blue, and $c(1,1)$ can be either red or yellow. 2 ways.

Case 2: $c(2,1)$ = yellow, then the neighbors of $c(3,1)$ are all yellow, so $c(3,1)$ = blue. $c(1,1)$ can be either red or blue. 2 ways.

Case 3: $c(2,1)$ = blue, then $c(1,1)$ = yellow or $c(3,1)$ = yellow.

However, neither has a neighbor with color red (contradiction!).

By shifting the colors, the total number of ways $= 3 \times (2+2) = 12$.

22 A seven-digit positive integer is chosen at random. What is the probability that the number is divisible by 11, given that the sum of its digits is 61?

A. $\frac{3}{14}$ B. $\frac{3}{11}$ C. $\frac{2}{7}$ D. $\frac{4}{11}$ E. $\frac{3}{7}$

Answer A

Solution Given $(x_1 \cdots x_7)_{10}$ with $\sum_{i=1}^{7} x_i = 61$ and $0 \le x_i \le 9$.

Let $y_i = 9 - x_i$, then $\sum_{i=1}^{7} y_i = 2$ and $0 \le y_i \le 9$.

This gives $\sum_{i=1}^{7}(y_i+1) = 9$, so there are $\binom{8}{6} = \binom{8}{2} = 28$ numbers.

From $\sum_{i=1}^{7} y_i = 2$, we have $y_i \le 2$ and $x_i \ge 7$.

Let $a = x_2 + x_4 + x_6$, then $x_1 + x_3 + x_5 + x_7 = 61 - a$.

For $11 \mid (x_1 \cdots x_7)_{10}$, we need $61 - a \equiv a \pmod{11}$, which gives $a \equiv 3 \pmod{11}$.

Since $a \ge 7 \times 3 = 21$ and $a \le 3 \times 9 = 27$, we must have $a = 11 \times 2 + 3 = 25$.

Then $61 - a = 36$, so $x_1 = x_3 = x_5 = x_7 = 9$ and $y_1 = y_3 = y_5 = y_7 = 0$.

For $y_2 + y_4 + y_6 = 2$ with $y_2, y_4, y_6 \ge 0$ and $y_i \in \mathbb{Z}$:

$(y_2+1) + (y_4+1) + (y_6+1) = 5$ has $\binom{5-1}{3-1} = 6$ solutions.

Therefore, there are 6 numbers in total, with probability $= \frac{6}{28} = \frac{3}{14}$.

23 A rectangular grid of souares has 141 rows and 91 columns. Each souare has room for two numbers. Horace and Vera each fill in the grid by putting the numbers from 1 through $41 \times 91 = 12,831$ into the squares. Horace fills the grid horizontally: he puts 1 through 91 in order from left to right into row 1, puts 92 through 182 into row 2 in order from left to right, and continues similarly through row 141. Vera fills the grid vertically: she puts 1 through 141 in order from top to bottom into column 1, then 142 through 282 into column 2 in order from top to bottom, and continues similarly through column 91. How many squares get two copies of the same number?

A. 7 B. 10 C. 11 D. 12 E. 19

Answer C

Solution Let a_{ij} and b_{ij} be the numbers filled by Horace and Vera respectively at row i, column j.

Then: $a_{ij} = 91(i-1)+j, \quad 1 \leq i \leq 141, \quad 1 \leq j \leq 91$

$b_{ij} = i + 141(j-1), \quad 1 \leq i \leq 91, \quad 1 \leq j \leq 141$

For $a_{ij} = b_{ij}$:

$91(i-1)+j = i + 141(j-1)$

Let $x = i-1$ and $y = j-1$:

$91x + y = x + 141y \iff 9x = 14y$

where $0 \leq x \leq 140$ and $0 \leq y \leq 90$.

This gives $x = 140t$ and $y = 9t$ for $0 \leq t \leq 10$.

There are 11 solutions.

24 A frog hops along the number line according to the following rules.

· It starts at 0.

· If it is at 0, then it moves to 1 with probability $\frac{1}{2}$ and it disappears with probability $\frac{1}{2}$.

· For $n = 1, 2$, or 3 , if it is at n, then it moves to $n+1$ with probability $\frac{1}{4}$, it moves to $n-1$ with probability $\frac{1}{4}$, and it disappears with probability $\frac{1}{2}$.

What is the probability that the frog reaches 4?

A. $\dfrac{1}{101}$ B. $\dfrac{1}{100}$ C. $\dfrac{1}{99}$ D. $\dfrac{1}{98}$ E. $\dfrac{1}{97}$

Answer E

Solution Let $P(i)$ be the probability that the frog reaches 4 starting at i, where $0 \leq i \leq 3$.

Then:

$$P(0) = \frac{1}{2}P(1)$$
$$P(1) = \frac{1}{4}P(2) + \frac{1}{4}P(0)$$
$$P(2) = \frac{1}{4}P(3) + \frac{1}{4}P(1)$$
$$P(3) = \frac{1}{4} + \frac{1}{4}P(2)$$

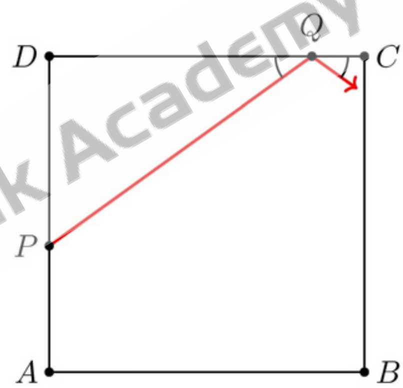
From the first equation: $P(1) = 2P(0)$

Plugging into the second equation: $P(2) = 7P(0)$

Solving the third equation: $P(3) = 26P(0)$

From the fourth equation: $4 \times 26P(0) = 1 + 7 \cdot P(0)$, which gives $P(0) = \dfrac{1}{97}$.

25 Square $ABCD$ has sides of length 4. Points P and Q lie on $\overline{AD}$ and $\overline{CD}$, respectively, with $AP = \dfrac{8}{5}$ and $DQ = \dfrac{10}{3}$. A path begins along the line segment from P to Q and continues by reflecting against the sides of $ABCD$ (with congruent incoming and outgoing angles), as shown in the figure. If the path hits a vertex of the square, then it terminates there; otherwise it continues forever.



At which vertex does the path terminate?

A. A

B. B

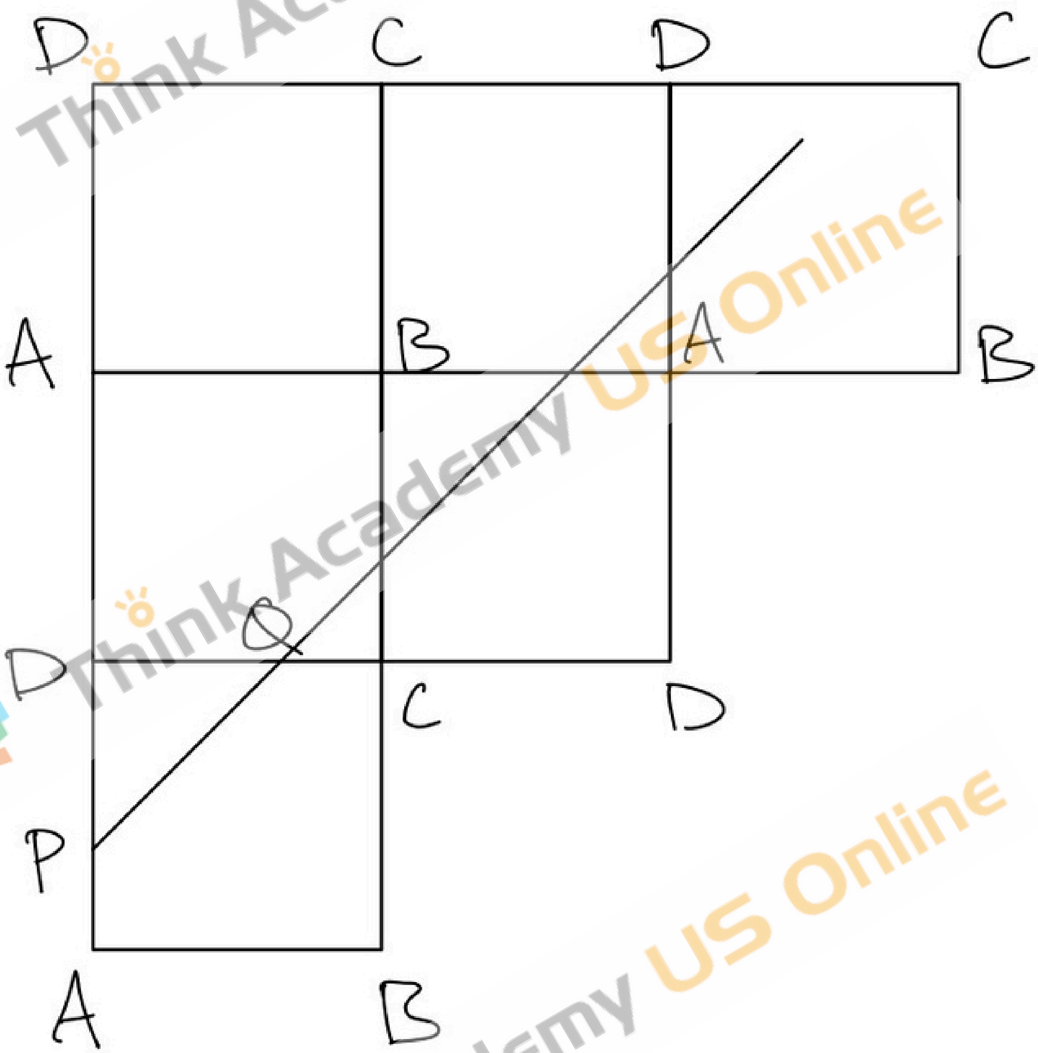
C. C

D. D

E. The path continues forever.

Answer B

Solution



We can let the line keep going, making the square reflected.

It suffices to check if line PQ passes $(4k, 4\ell)$ for some $(k, \ell) \in \mathbb{N}_+$.

Let line PQ: $y = mx + \frac{8}{5}$, then $m = \frac{4 - \frac{8}{5}}{\frac{10}{3}} = \frac{18}{25}$.

So PQ: $y = \frac{18}{25}x + \frac{8}{5}$.

For $x = 4k$ and $y = 4\ell$: $25\ell = 18k + 10$.

The solution $(k, \ell) = (5, 4)$ is a positive integer solution.

It can be proved that $(8k, 8\ell)$ are images of A, $(8k+4, 8\ell)$ are images of B, $(8k, 8\ell+4)$ are images of D, and $(8k+4, 8\ell+4)$ are images of C.

Therefore $(20, 16)$ is an image of B.