AMC Day19

1 （1分）The region consisting of all points in three-dimensional space within 3 units of line segment $\overline{AB}$ has volume 216π. What is the length AB? （2017 AMC 10A Problem 11）

A. 6 B. 12 C. 18 D. 20 E. 24

Answer D

Solution The region is a cylinder of radius 3 with a hemisphere of radius 3 at each end. The two hemispheres together form a sphere of radius 3.

Let $x = AB$, which is the height of the cylinder. The total volume is $\pi \cdot 3^2 x + \frac{4}{3}\pi \cdot 3^3 = 216\pi.$

Thus, $9\pi x + 36\pi = 216\pi, 9x = 180, \quad x = 20.$

Therefore, the length AB is $\boxed{(\mathrm{D})\ 20}$.

2 （1分）A rectangular box $\mathcal{P}$ has distinct edge lengths a, b, and c. The sum of the lengths of all 12 edges of $\mathcal{P}$ is 13, the sum of the areas of all 6 faces of $\mathcal{P}$ is $\frac{11}{2}$, and the volume of $\mathcal{P}$ is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of $\mathcal{P}$?
（2023 AMC 10B Problem 17）

A. 2 B. $\frac{3}{8}$ C. $\frac{9}{8}$ D. $\frac{9}{4}$ E. $\frac{3}{2}$

Answer D

Solution The longest diagonal passes through the interior of the box, connecting opposite vertices; its length is $\sqrt{a^2+b^2+c^2}$.

Because $4a+4b+4c=13 \quad \text{and} \quad 2ab+2bc+2ac=\frac{11}{2}$, it follows that

$$\begin{aligned}\left(\frac{13}{4}\right)^2 &= (a+b+c)^2 \\ &= a^2+b^2+c^2+2ab+2bc+2ac \\ &= a^2+b^2+c^2+\frac{11}{2}.\end{aligned}$$

Therefore, $a^2+b^2+c^2=\left(\frac{13}{4}\right)^2-\frac{11}{2}=\frac{81}{16}$. The longest diagonal has length $\sqrt{a^2+b^2+c^2}=\sqrt{\frac{81}{16}}=\boxed{\text{(D) } \frac{9}{4}}$.

Note: Such a box exists with edge lengths $\frac{1}{4}$, 1, and 2. The volume is not needed to solve this problem.

3 （1分）Let B be a right rectangular prism (box) with edge lengths 1, 3, and 4, together with its interior. For real $r \geq 0$, let $S(r)$ be the set of points in 3-dimensional space that lie within a distance r of some point in B. The volume of $S(r)$ can be expressed as ar^3+br^2+cr+d, where a, b, c, and d are positive real numbers. What is $\frac{bc}{ad}$? （2020 AMC 10B Problem 20）

A. 6　　B. 19　　C. 24　　D. 26　　E. 38

Answer B

Solution We divide $S(r)$ into the original box and the regions extending outward from its faces, edges, and vertices.

The original box has volume $1 \cdot 3 \cdot 4 = 12$.

Each face contributes a rectangular prism of thickness r. Their total volume is the surface area of the box multiplied by r: $2(1 \cdot 3 + 1 \cdot 4 + 3 \cdot 4)r = 38r$.

Each edge contributes a quarter-cylinder of radius r, with length equal to that edge. Since the total length of all 12 edges is $4(1+3+4)=32$, these quarter-cylinders have total volume $\frac{1}{4}\pi r^2 \cdot 32 = 8\pi r^2$.

Each of the 8 vertices contributes one eighth of a sphere of radius r. Together, these pieces form a complete sphere with volume $\frac{4}{3}\pi r^3$.

Therefore, $V(S(r)) = \frac{4}{3}\pi r^3 + 8\pi r^2 + 38r + 12$.

Hence, $a=\frac{4\pi}{3}$, $b=8\pi$, $c=38$, $d=12$,

and $\frac{bc}{ad} = \frac{8\pi \cdot 38}{\frac{4\pi}{3} \cdot 12} = \boxed{\text{(B) } 19}$.