$5 \rightarrow 10, 11, 12$ (let's go with 12)

$4 \rightarrow 8, 9, 10, 11$ (let's go with 11)

$3 \rightarrow 8, 9, 10$ (let's go with 10)

$2 \rightarrow 8, 9$ (let's go with 9)

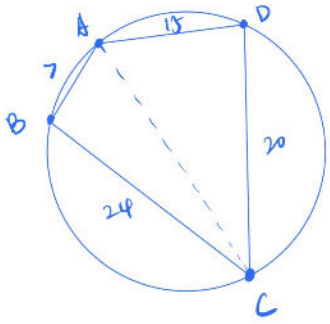
$1 \rightarrow 8$

15 Quadrilateral $ABCD$ with side lengths $AB = 7$, $BC = 24$, $CD = 20$, $DA = 15$ is inscribed in a circle. The area interior to the circle but exterior to the quadrilateral can be written in the form $\frac{a\pi - b}{c}$, where a, b, and c are positive integers such that a and c have no common prime factor. What is $a + b + c$?

A. 260 B. 855 C. 1235 D. 1565 E. 1997

Answer D

Solution



$7^2 + 24^2 = 25^2 = 15^2 + 20^2$

$\angle B + \angle D = 180°$

If $\angle B > 90°, AC > 75$

$\because \angle B + \angle D = 180°$

$\therefore \angle D < 90°, AC < 25$, so there is a contradiction.

If $\angle B < 90°$, we also have a contradiction.

$\therefore \angle B = \angle D = 90°$,

$\therefore$ AC is the diameter of the circle.

$\therefore A_{ABCD} = A_{ABC} + A_{ACD} = \frac{1}{2} \times 7 \times 24 + \frac{1}{2} \times 15 \times 20 = 234$,

$$A_{circle} = \pi \cdot \left(\frac{25}{2}\right)^2 = \frac{625\pi}{4}.$$
$$\therefore \frac{a\pi - b}{c} = \frac{625\pi}{4} - 234 = \frac{625\pi - 936}{4}$$
$$\therefore a + b + c = 625 + 936 + 4 = 1565.$$

16 The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

A. $\frac{24}{5}$ B. $\frac{42}{5}$ C. $\frac{81}{5}$ D. 30 E. 48

Answer D

Solution Assume the roots of the polynomial are a, b, c.

According to Vieta's formula,

$$\begin{cases} a + b + c = \frac{39}{10} \\ ab + bc + ca = \frac{29}{10} \\ abc = \frac{3}{5} \end{cases}$$

$$(a+2)(b+2)(c+2) = abc + 2(ab + bc + ca) + 4(a + b + c) + 8 = 30.$$

17 How many three-digit positive integers $\underline{a}\,\underline{b}\,\underline{c}$ are there whose nonzero digits a, b, and c satisfy

$$0.\overline{\underline{a}\,\underline{b}\,\underline{c}} = \frac{1}{3}(0.\overline{a} + 0.\overline{b} + 0.\overline{c})?$$

(The bar indicates repetition, thus $0.\overline{\underline{a}\,\underline{b}\,\underline{c}}$ is the infinite repeating decimal $0.\underline{a}\,\underline{b}\,\underline{c}\,\underline{a}\,\underline{b}\,\underline{c}\cdots$)

A. 9 B. 10 C. 11 D. 13 E. 14

Answer D

Solution $0.\overline{\underline{a}\,\underline{b}\,\underline{c}}$ is a recuesive decimal, so as $0.\overline{a}$, $0.\overline{b}$ and $0.\overline{c}$.

$$\frac{100a + 10b + c}{999} = \frac{1}{3} \cdot \left(\frac{a}{9} + \frac{b}{9} + \frac{c}{9}\right)$$
$$\frac{100a + 10b + c}{999} = \frac{a + b + c}{27}$$
$$100a + 10b + c = 37a + 37b + 37c$$

$63a = 27b + 36c$

$3b + 4c = 7a$

$\therefore b \equiv c \pmod 7$.

Thus, either $b = c$ or $(b, c) = (1, 8), (2, 9), (9, 2)$. Each of these will have an unique solution, yielding the 13 intergers: $111, 222, 333, 444, 555, 666, 777, 888, 999, 518, 629, 481, 592$.

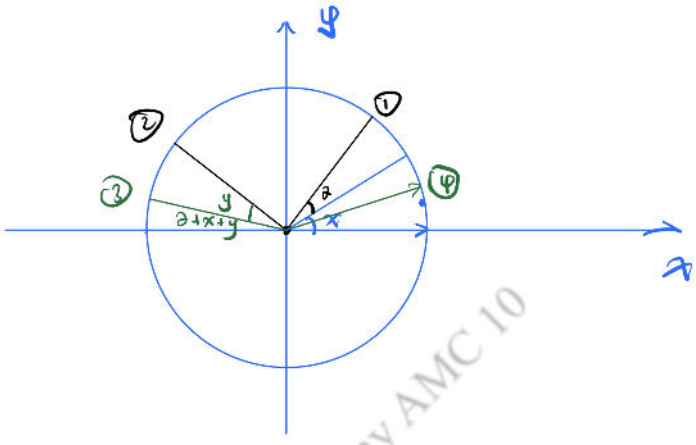
Therefore, the answer is 13.

18 Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \ldots, T_n$ returns the point $(1, 0)$ back to itself?

A. 359 B. 360 C. 719 D. 720 E. 721

Answer A

Solution



Let the angle between 'the point', 'the origin' and the positive x-axis be α.

$T_x : \alpha \to \alpha + x \to 180 - (\alpha + x)$

$T_x + T_y : 180 - (\alpha + x) \to 180 - (\alpha + x) + y \to \alpha + x - y$

Initially: $0°$ after $T_1, T_2 \ldots T_n$.

When $n = 2k$, the angle:

$(1 - 2) + (2 - 3) + \ldots [(2k + 1) - 2k] = -k$.

Back to the original position: $k = 360°$, $n = 720$.