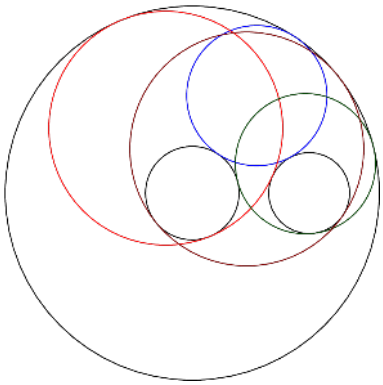
Solution



There are two circles that are externally tangent to the two small circles and internally tangent to the large circle. Here $r = 3$ and $A = 18\pi$. (blue part)

There are two circles that are internally tangent to the concentric circles and externally tangent to the other circle. Here $r = 5$ and $A = 50\pi$. (red part)

There are two circles that are internally tangent to the big and non-center circles, but externally tangent to the small center circle. Here $r = 3$ and $A = 18\pi$. (green part)

There are two circles that are internally tangent to all three circles. Here $r = 5$ and $A = 50\pi$. (brown part)

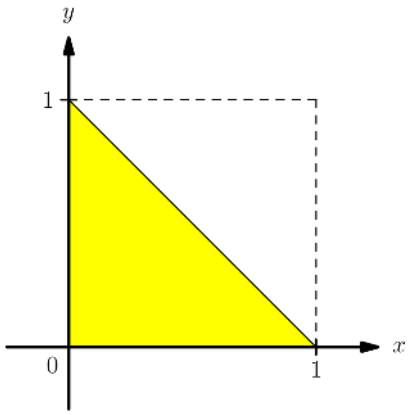
The sum of area of all circles is 136π.

23 Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n = 1, 2, 3$, Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0, 1)$. During the nth step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the n^{th} step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

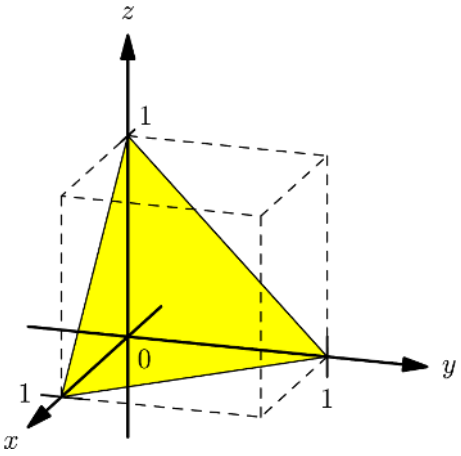
A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. $\frac{2}{3}$ D. $\frac{3}{4}$ E. $\frac{5}{6}$

Answer C

Solution Let x and y be random variables that are independently and uniformly distributed in the interval $(0, 1)$. Note that $P(x + y \leq 1) = \frac{\frac{1}{2} \cdot 1^2}{1^2} = \frac{1}{2}$, as shown below:



Let x, y, and z be random variables that are independently and uniformly distributed in the interval $(0,1)$. Note that $P(x+y+z \leq 1) = \frac{\frac{1}{3} \cdot (\frac{1}{2} \cdot 1^2) \cdot 1}{1^3} = \frac{1}{6}$, as shown below:



We have two cases: 1. Amelia takes exactly 2 steps. We need $x_1 + x_2 > 1$ and $t_1 + t_2 > 1$. So, the probability is $P(x_1 + x_2 > 1) \cdot P(t_1 + t_2 > 1) = \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{2}\right) = \frac{1}{4}$.

2. Amelia takes exactly 3 steps. We need $x_1 + x_2 + x_3 > 1$ and $t_1 + t_2 \leq 1$. So, the probability is $P(x_1 + x_2 + x_3 > 1) \cdot P(t_1 + t_2 \leq 1) = \left(1 - \frac{1}{6}\right) \cdot \frac{1}{2} = \frac{5}{12}$. Together, the answer is $\frac{1}{4} + \frac{5}{12} = (\text{C})\ \frac{2}{3}$.

24 Consider functions f that satisfy $|f(x) - f(y)| \leqslant \frac{1}{2}|x - y|$ for all real numbers x and y. Of all such functions that also satisfy the equation $f(300) = f(900)$, what is the greatest possible value of

$$f(f(800)) - f(f(400))?$$

A. 25 B. 50 C. 100 D. 150 E. 200

Answer B

Solution $|f(800)-f(900)| \leqslant \frac{1}{2}|800-900| = 50;$

$|f(400)-f(300)| \leqslant \frac{1}{2}|400-300| = 50.$

$|f(800)-f(400)| = |f(800-f(900)+f(300)-f(400)|$

$\leqslant |f(800)-f(900)| + |f(300)-f(400)|$

$\leqslant 50+50 = 100.$

$$\begin{aligned} f[f(800)]-f[f(400)] &\leqslant \frac{1}{2}|f(800)-f(400)| \\ &\leqslant \frac{1}{2}\times 100 \\ &= 40 \end{aligned}$$

25 Let $x_0, x_1, x_2 \cdots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k$$

Suppose $7S_n = 1 \pmod{2^n}$ for all $n \geqslant 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

A. 6 B. 7 C. 12 D. 14 E. 15

Answer A

Solution $7S_1 = 3\times 2^1 + 1 \Rightarrow x_0 = 1$
$7S_2 = 5\times 2^2 + 1 \Rightarrow x_1 = 1$
$7S_3 = 6\times 2^3 + 1 \Rightarrow x_2 = 1$
$7S_4 = 3\times 2^4 + 1 \Rightarrow x_3 = 0$
$7S_5 = 5\times 2^5 + 1 \Rightarrow x_4 = 1$
$7S_6 = 6\times 2^6 + 1 \Rightarrow x_5 = 1$
$7S_7 = 3\times 2 + 1 \Rightarrow x_6 = 0$
$\cdots$

With the patterns, we notice that $x_{3n} = 0$ and $x_{3n+1} = x_{3n+2} = 1$. Therefore,

$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022} = 0 = 2 + 4 + 0 = 6.$