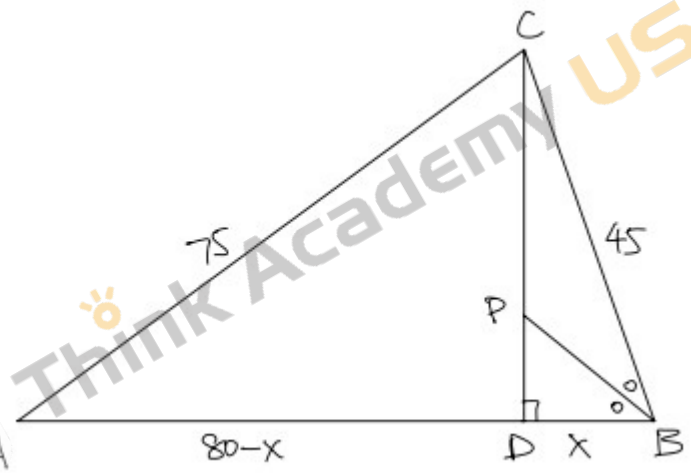
Answer D

Solution



Set $BD = x$, making $AD = 80 - x$. $BC^2 - BD^2 = CD^2 = AC^2 - AD^2$

This gives $45^2 - x^2 = 75^2 - (80 - x)^2$.

Solving: $x = \frac{35}{2}$, and $CD^2 = 45^2 - \left(\frac{35}{2}\right)^2 = \frac{25 \times 275}{4}$, so $CD = \frac{25\sqrt{11}}{2}$.

By the angle bisector theorem, $\frac{DP}{PC} = \frac{BD}{BC} = \frac{7}{18}$

So $DP = \frac{7}{18+7} \times CD = \frac{7\sqrt{11}}{2}$

Therefore: $BP^2 = BD^2 + DP^2 = 441$

Thus $BP = 21$.

24 Call a positive integer ***fair*** if no digit is used more than once, it has no 0s, and no digit ins adjacent to two greater digits. For example, 196, 23, and 12463 are fair, but 1546, 320, and 34321 are not fair. How many fair positive integers are there?

A. 511 B. 2584 C. 9841 D. 17711 E. 19682

Answer C

Solution For k selected digits, let a_k count arrangements where no digit is adjacent to two larger digits.

Base cases: $a_1 = 1, a_2 = 2$.

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For $k \geq 3$, the smallest digit must occupy either the first or last position.

Each placement corresponds bijectively to arranging $k-1$ digits with the same constraint, giving $a_k = 2a_{k-1}$, hence $a_k = 2^{k-1}$.

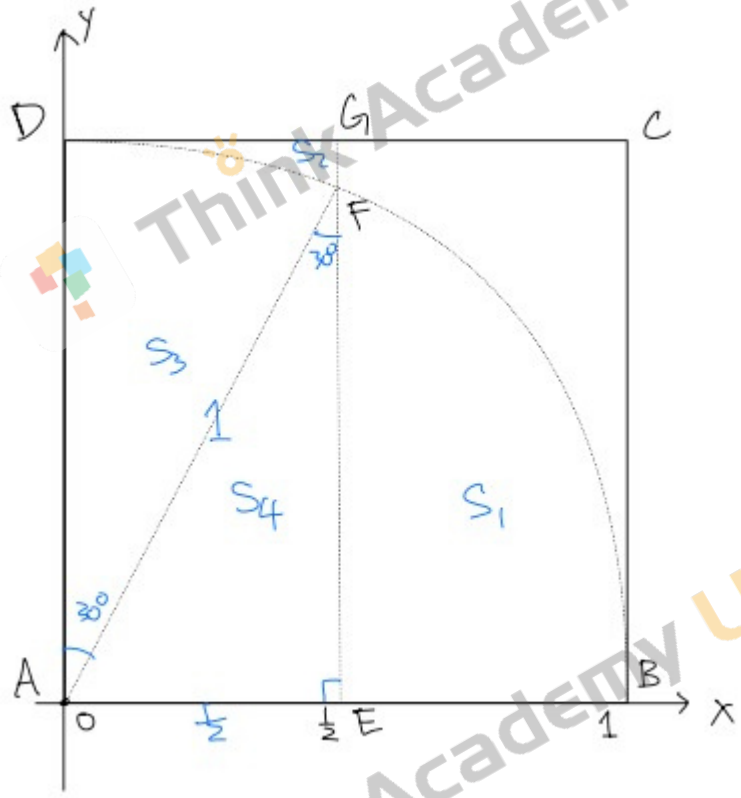
Total fair integers: $\sum_{k=1}^{9} \binom{9}{k} \cdot 2^{k-1} = \frac{\sum_{k=0}^{9} \binom{9}{k} \cdot 2^k - 1}{2} = \frac{(2+1)^9 - 1}{2} = \frac{3^9 - 1}{2} = 9841$

25 A point P is chosen at random inside square $ABCD$. The probability that $\overline{AP}$ is neither the shortest nor the longest side of $\triangle APB$ can be written as $\frac{a + b\pi - c\sqrt{d}}{e}$, where a, b, c, d, and e are positive integers, $gcd(a, b, c, e) = 1$, and d is not divisible by the square of any prime. What is $a + b + c + d + e$?

A. 25 B. 26 C. 27 D. 28 E. 29

Answer A

Solution



Establish coordinates: $A(0,0), B(1,0), C(1,1), D(0,1)$.

Scenario 1: $AB > AP > BP$. For point $P(x, y)$:

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Condition $AP > BP$ implies $x > \frac{1}{2}$, and condition $AB > AP$ implies $AP < 1$.

Point P lies within the unit circle centered at A and to the right of $x = \frac{1}{2}$ which is EF. This defines region S_1.

Scenario 2: $AB < AP < BP$.

Condition $AP < BP$ implies $x < \frac{1}{2}$, and condition $AB < AP$ implies $AP > 1$.

Point P lies outside the unit circle and to the left of $x = \frac{1}{2}$. This defines region S_2.

Mark S_3 and S_4 as in the diagram too. Notice that $\angle OEF = 90^\circ$ and $\frac{OE}{OF} = \frac{1}{2}$. So $\angle OFE = \angle DAF = 30^\circ$

Thus $S_3 = \frac{30^\circ}{360^\circ}\pi \cdot 1^2 = \frac{\pi}{12}$, and $S_4 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$.

Also $S_2 + S_3 + S_4 = A_{DGEA} = \frac{1}{2}$, and $S_1 + S_3 + S_4 = \frac{90^\circ}{360^\circ}\pi \cdot 1^2 = \frac{\pi}{4}$.

Final result: $S_1 + S_2 = \frac{6 + \pi - 3\sqrt{3}}{12}$.

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