Solution For the general case with n students and n tutors:

Let S_i be the case where the i-th student has the same tutor, then $|S_i| = (n-1)!$ and

$|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}| = (n-k)!$.

$|(S_1 \cup \cdots \cup S_n)^c| = n! - |S_1 \cup \cdots \cup S_n|$

$= n! - \left(\sum_{i=1}^{n} |S_i| - \sum_{1 \le i_1 < i_2 \le n} |S_{i_1} \cap S_{i_2}| + \cdots + (-1)^{n+1} |S_1 \cap \cdots \cap S_n| \right)$

$= n! - \left(n \times (n-1)! - \binom{n}{2} \cdot (n-2)! + \cdots + (-1)^{n+1} \cdot 1 \right)$

The k-th term inside the bracket equals $(-1)^{k+1} \cdot \binom{n}{k} \cdot (n-k)! = (-1)^{k+1} \cdot \frac{n!}{k!}$.

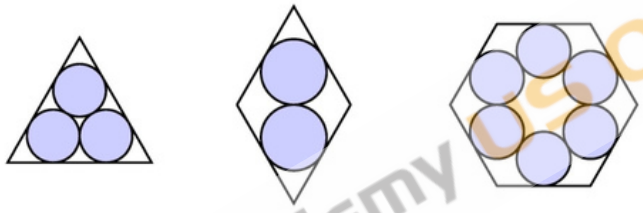
This gives $n! - \sum_{k=1}^{n} (-1)^{k+1} \cdot \frac{n!}{k!} = n! \cdot \left(\sum_{k=0}^{n} (-1)^k \cdot \frac{1}{k!} \right)$

For any 2 students, the total ways of assigning so that they get the same tutor, are the total ways of assigning the remaining 4 tutors so no students among these 4 get the same tutor:

$4! \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right) = 9$

Therefore, the probability $= \frac{\binom{6}{2} \times 9}{6!} = \frac{3}{16}$

12 The figure below shows an equilateral triangle, a rhombus with a 60° angle, and a regular hexagon, each of them containing some mutually tangent congruent disks. Let T, R, and H, respectively, denote the ratio in each case of the total area of the disks to the area of the enclosing polygon.



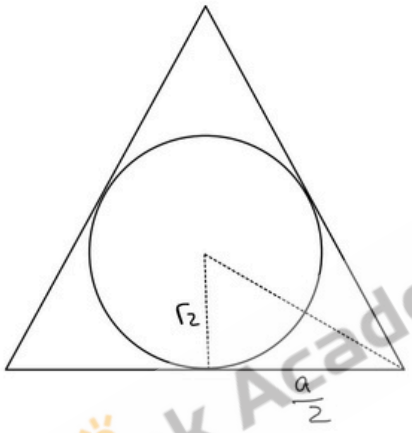
Which of the following is true?

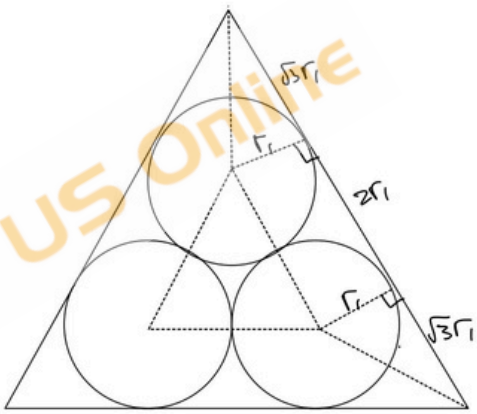
A. $T = R = H$ B. $H < R = T$ C. $H = R < T$ D. $H < R < T$ E. $H < T < R$

Answer C

Solution

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Let a be the length of a side of an equilateral triangle, r_1 be the radius in the 3-tangent-circle case, and r_2 be the one-tangent-circle case.

It can be checked that both R and H equal the ratio of the r_2 case.

Then: $a = 2\sqrt{3}r_1 + 2r_1 = 2(\sqrt{3}+1)r_1$ $a = 2\sqrt{3}r_2$

$$\frac{T}{R} = \frac{3\pi r_1^2}{\pi r_2^2} = \frac{3 \times 3}{(\sqrt{3}+1)^2} = \frac{9}{4+2\sqrt{3}} > 1$$

Thus $T > R = H$.

13 The altitude to the hypotenuse of a $30-60-90°$ right triangle is divided into two segment of lengths $x < y$ by the median to the shortest side of the triangle. What is the ratio $\frac{x}{x+y}$?

A. $\frac{3}{7}$ B. $\frac{\sqrt{3}}{4}$ C. $\frac{4}{9}$ D. $\frac{5}{11}$ E. $\frac{4\sqrt{3}}{15}$

Answer A

Solution

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