AMC10A soln

P1: B

Helena needs $33 \times 3 - 45 = 54$ more EP records.

P2: A

without tax : 100% tax : 27%

total price : 127%

So the percentage is $27/127 = 21.2\% \approx \boxed{21\%}$

P3: D

$\because \frac{AF}{FE} = \frac{AD}{BE} = 2$

$\therefore [BEF] = \frac{1}{3}[ABE] = \frac{1}{12}[ABCD] = \boxed{48}$

P4: A

Assume that it is a n-gon.

Then the sum of internal angles is : $(n-2) \cdot 180^\circ$,

which is also $\frac{n}{2} \cdot 150^\circ + \frac{n}{2} \cdot 190^\circ$.

Thus $n = \boxed{36}$

P5: C

Let the average of the first n terms be $A_n = \frac{a_1+a_2+\cdots+a_n}{n}$.

For $n \geq 2$, we have $n+1 \geq 3$, $a_{n+1} = A_n$.

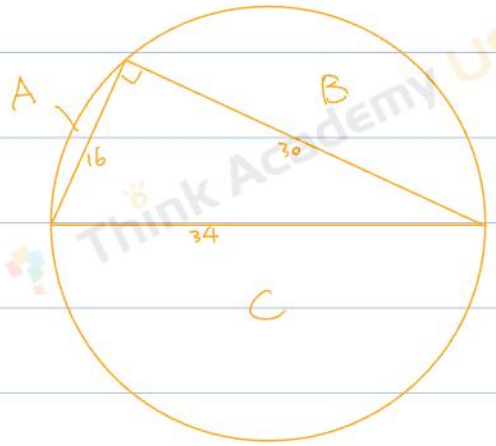
Thus $A_{n+1} = \frac{(a_1+a_2+\cdots+a_n)+a_{n+1}}{n+1} = \frac{(n\cdot A_n)+A_n}{n+1} = A_n$.

So $a_{10} = A_9 = A_8 = \cdots = A_2 = \boxed{142}$

P6: A

Notice that $16^2 = 4\times 64 = (34-30)(34+30) = 34^2 - 30^2$

$16^2 + 30^2 = 34^2$, it is a right triangle.



Thus the area of A+B is:

$\frac{1}{2}\cdot\pi\left(\frac{34}{2}\right)^2 - \frac{1}{2}\cdot 16\cdot 30 = \boxed{\frac{289}{2}\pi - 240}$

P7: A

Mark the expressions as ①, ② respectively.

So $y+z = \frac{②-①}{2} = 5$

$x = ① - (y+z) = 26$

There are $\boxed{4}$ solutions of y, z whereas x is fixed.

P8 : C

To have the lowest possible speed, we want shortest distance and longest time. Assume the next exit is 6.5 miles and 5.5 minutes away.

So the lowest speed is $\frac{6.5 \text{ miles}}{5.5 \text{ minutes}} \cdot 60 \text{ minutes/hour} \approx 70.9 \text{ mph}$.

The lowest speed limit is $\boxed{75}$ mph.

P9 : D

First assume the longest side of the larger triangle > 12.

If it turns out to be impossible, the longest side is then 12.

Since the two similar scalene triangles are not congruent, their shortest, middle, longest sides are not equal. Thus:

~~side~~ triangle	a	$<$	b	$<$	c
larger	9		12		x
smaller	y		9		12

Then $\frac{9}{y} = \frac{12}{9} = \frac{x}{12}$, $\begin{cases} x = \boxed{16} \\ y = \frac{27}{4} \end{cases}$.

P10 : C

The mean is $\frac{1^2+2^2+\cdots+n^2}{n} = \left[\frac{1}{6}n(n+1)(2n+1)\right]/n = \frac{(n+1)(2n+1)}{6}$

The median for odd n is : $\left(\frac{n+1}{2}\right)^2$

So $\frac{1}{6}(n+1)(2n+1) = (1+32\%)\left(\frac{n+1}{2}\right)^2$

$\Longrightarrow \quad n = \boxed{49}$

P11 : D

We know that the largest number, also the lcm of all six numbers, is the multiple of the other five numbers.

Furthermore, notice that $2, 5, 24, 60 \mid 240$.

Case 1: $n > 240$ is the largest number.

Then $240 \mid n$, $n = 480, 720, 960$.

Case 2: $240 > n$ is the largest number.

Then $n \mid 240$, the three-digit number $n = 120$.

So the sum is $480 + 720 + 960 + 120 = \boxed{2280}$.

P12 : D

Let Pippi keep removing socks until all are removed.

If there are 2 socks (red) left when all socks of the other color (blue) are removed, we know the removing order is:

$\underbrace{\qquad\qquad}$, blue, red, red.

freely order 98 red socks,
99 blue socks.

$\Rightarrow$ There are $\binom{197}{98}$ ways to achieve the desired scenario.

Similarly, when all red socks are removed first,
there are $\binom{197}{98}$ ways.

Otherwise we are freely ordering 100 red socks, 100 blue socks,
with $\binom{200}{100}$ total ways.

So the probability is $\frac{2\cdot\binom{197}{98}}{\binom{200}{100}} = \frac{2\cdot\frac{197!}{98!\,99!}}{\frac{200!}{100!\,100!}} = \boxed{\frac{50}{199}}$.

P13 : C

Let there be p and q people in two clubs. Assume $p \le q$.

Thus p, q are positive integers.

Then $\begin{cases} 119 = \binom{p}{2} + \binom{q}{2} = \frac{1}{2}\left(p(p-1) + q(q-1)\right) \\ 112 = pq \end{cases}$

Factorize $112 = 2^4 \cdot 7 = 1 \times 112$

$= 2 \times 56$

$= 4 \times 28$

$= 7 \times 16$

$= 8 \times 14$

Only $\begin{cases} p = 8 \\ q = 14 \end{cases}$ satisfies both conditions.

So $q - p = \boxed{6}$.

P14 : B

By triangular inequality, we have:

$$t^2 + 1 = a < b + c = (5t + 7) + (4t + 4) = 9t + 11$$

Thus $t^2 - 9t - 10 = (t - 10)(t + 1) < 0$,

$-1 < t < 10$.

Check $t \in \{0, 1, \cdots, 9\}$, we see that only $t \in \{3, 4, \cdots, 9\}$ are desired, there are $\boxed{7}$ possible integer t.

P15: B

Tracing backwards, if $\frac{a}{b}$'s next level is $\frac{20}{27}$,

then $\frac{20}{27} = \frac{a}{a+b}$ or $\frac{b}{a+b} \Longrightarrow \begin{cases} a = 20, & 7 \\ b = 7, & 20. \end{cases}$

Since every fraction on the tree is either the top $\frac{1}{2}$, or generated as $\frac{a}{a+b}, \frac{b}{a+b}$, all of them are in $(0,1)$.

Thus it can only be $\frac{7}{20}$ that generated $\frac{20}{27}$.

Repeat the tracing backwards:

$$\frac{20}{27} \leftarrow \frac{7}{20} \leftarrow \frac{7}{13} \leftarrow \frac{6}{7} \leftarrow \frac{1}{6} \leftarrow \frac{1}{5}$$

$$\leftarrow \frac{1}{4} \leftarrow \frac{1}{3} \leftarrow \frac{1}{2} \text{ (level 0)}.$$

So $\frac{20}{27}$ is at level $\boxed{8}$.

P16: D



Notice that the minimum score is 4, with possible

		red, black, white		
score	counts	(3, 1, 0)	(3, 0, 1) and	(0, 2, 2);
		↕	↕	↕
ball	counts	(6, 1, 0)	(6, 0, 1) and	(0, 4, 3)

$(0, 3, 4)$.

$\Rightarrow$ number of ways to get lowest score:

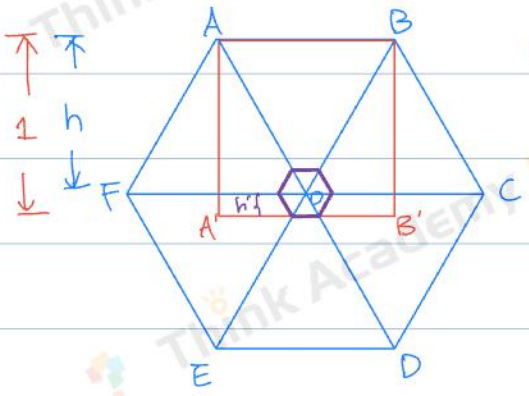
$$\binom{6}{6}\binom{4}{1} + \binom{6}{6}\binom{4}{1} + \binom{4}{4}\binom{4}{3} + \binom{4}{3}\binom{4}{4} = 16$$

number of total ways to choose 7 balls out of $6+4+4=14$ balls:

$$\binom{14}{7} = 3432.$$

The probability is $16/3432 = \boxed{2/429}$

P17: C



Consider the top square $ABB'A'$.

A part of side $A'B'$ becomes a side of the small hexagon.

Since $h = \frac{\sqrt{3}}{2}$, we have $h' = 1 - h = 1 - \frac{\sqrt{3}}{2}$.

So the area of the small hexagon, which is similar to $ABCDEF$, is

$$[ABCDEF] \cdot \left(\frac{h'}{h}\right)^2 = 6 \cdot \frac{1}{2}\left(1 \cdot \frac{\sqrt{3}}{2}\right) \cdot \left(\frac{1-\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}\right)^2$$

$$= \frac{3\sqrt{3}}{2} \cdot \left(\frac{2}{\sqrt{3}} - 1\right)^2 = \frac{3\sqrt{3}}{2}\left(\frac{7}{3} - \frac{4\sqrt{3}}{3}\right)$$

$$= \frac{7\sqrt{3}}{2} - 6$$

Thus $a+b+c+d = 7+3+2+6 = \boxed{18}$

P18: D

For $k \in \{1, 2, \cdots, 10\}$, $a_k = \frac{1}{2}a_{k-1} + 2 < a_{k-1}$,

thus $a_{k-1} > 4$

So $a_0, a_1, \cdots, a_9 > 4$.

Notice that $a_{k-1} = 2a_k - 4$

To minimize $a_0 = 2a_1 - 4$, we need to minimize a_1, and thus $a_2, a_3, \cdots, a_9$. If $a_9 = 5 > 4$, $a_{10} = \frac{1}{2} \cdot 5 + 2$ is not an integer.

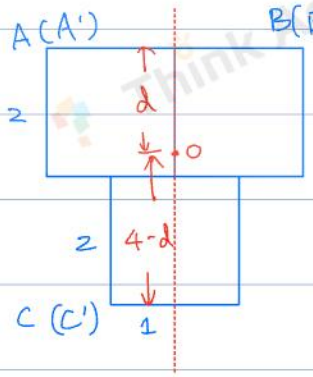
Let $a_9 = 6$, then we compute $a_0 \sim a_8$:

n	9	8	7	6	5	4	3	2	1	0
a_n	6	8	12	20	36	68	132	260	516	1028

(You may prove $a_n = 2^{10-n} + 4$ by induction)

So the sum of digits is: $1+0+2+8 = \boxed{11}$

P19: C



Look at top view of the diagram,

By reflection symmetry, assume the sphere center O is on the symmetric axis.

Let d be the distance to AB from O.

A, B, C are cube vertices on the table surface.

Consider the cube vertices 2 units above A, B, C to be A', B', C' respectively. We need radius $r \geq OA' = OB', OC'$.

$$r^2 \geq \max(OA'^2, OC'^2).$$

Since $OA'^2 = OB'^2 = OA^2 + 2^2 = (2^2 + d^2) + 2^2 = d^2 + 8$

$OC'^2 = OC^2 + 2^2 = (1^2 + (4-d)^2) + 2^2 = (4-d)^2 + 5$,

to minimize r^2, we need to let $OA'^2 = d^2 + 8 = (4-d)^2 + 5 = OC'^2$.

Then $d = \frac{13}{8}$, $OA'^2 = OC'^2 = \frac{681}{64}$.

Thus $r_{min} = \sqrt{\frac{681}{64}} = \boxed{\frac{\sqrt{681}}{8}}$.

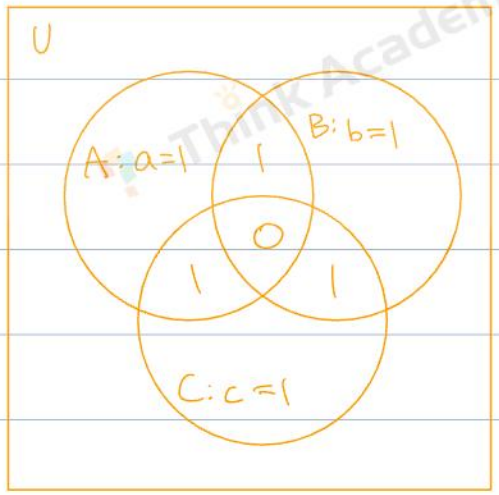
P20: C

Factorize $510510 = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17$

$= a \times b \times c$

where a, b, c are the sides of a box.

So we are assigning the 7 distinct prime factors into 3 slots a, b, c.



Consider the Venn diagram, region A, B, C satisfies a=1, b=1, c=1 respectively.

If we freely assign the factors, at least one of $a, b, c \neq 1$. Thus $A \cap B \cap C = \emptyset$.

In $A \cap B$, $a=b=1$, so $c = 5!0^5!0$. Thus

$$|A \cap B| = |B \cap C| = |C \cap A| = 1$$

In A, $a=1$, so we need to assign each of 7 factors to b or c.

Thus $|A| = |B| = |C| = 2^7$.

In U, the total ways to make $5!0^5!0 = a \times b \times c$, we shall assign each of 7 factors to a or b or c. Thus $|U| = 3^7$.

So for $a, b, c \geq 2$, or equivalently $a, b, c \neq 1$, we have

$$|U - A \cup B \cup C| = 3^7 - (3 \times 2^7 - 3 \times 1 + 0)$$

$$= 1806$$

Notice that a, b, c must be distinct, and all six permutations of $a \times b \times c$:

$a \times b \times c = a \times c \times b = b \times a \times c = b \times c \times a = c \times a \times b = c \times b \times a$

are considered the same, so we have $1806/6 = \boxed{301}$ ways.

P21: C

Consider how many "a" squares is in the section of our central square.

b	a_2	b
a_1		a_3
b	a_4	b

Case 1: two "a". There are 2 legit sections:

$a_1 a_3$ or $a_2 a_4$.

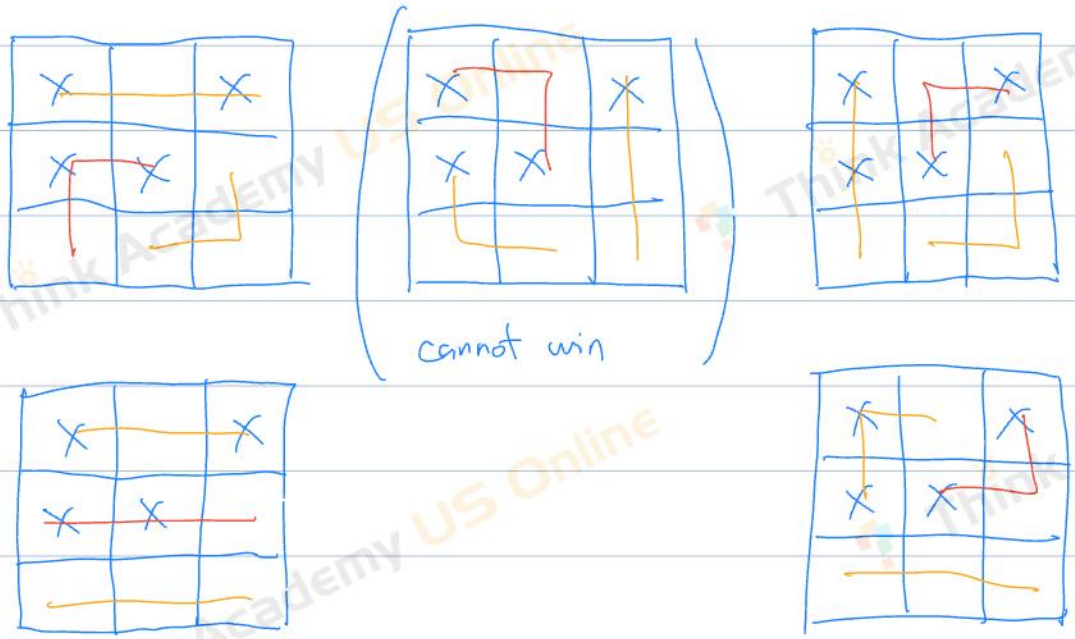
Case 2: one "a", then there is also one "b" in the partition. We have

4 choices for "a" × 2 choices for "b"

= 8 legit sections.

After the section of the central square is determined, the whole partition is also determined, so we have 10 partitions.

Now consider the winning partition. We must have two "2×10" section. Let's choose the second "x" for our central "x".

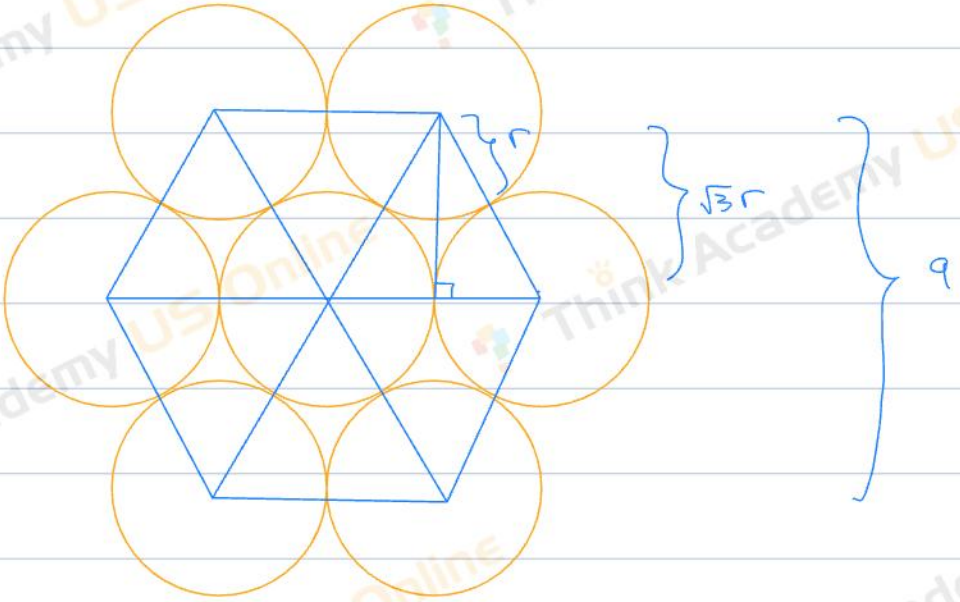


By finishing the partitions, we see that there are 4 desired.

The probability is $4/10 = \boxed{2/5}$.

P22: A

Imagine a horizontal dilation that stretches the ellipses into circles. The vertical distance isn't changed.



Thus $2\sqrt{3}r = q$, $r = \frac{3\sqrt{3}}{2}$,

the vertical major axis is unchanged, which is $2r = \boxed{3\sqrt{3}}$.

P23: D

Let m, n be the first term of the list.

Conside $m, n \bmod 6 = 2 \times 3$, and the smallest (p, q) pair in each condition. Consider $m \equiv n \pmod 6$ then fill in the rest.

m \ n	0 (no change)	1 (p+1)	2 (q+1)	3 (p+1)	4 (no change)	5 (p, q+1)
0	~~(1,1)~~	(1,1)	(2,1)	(2,2)	(3,2)	(3,2)
1	(3,2)	~~(0,0)~~	(1,0)	(1,1)	(2,1)	(2,1)

2	(3,2)	(3,2)	(1,0)	(1,1)	(2,1)	(2,1)
3	(2,2)	(2,2)	(3,2)	(0,1)	(1,1)	(1,1)
4	(2,1)	(2,1)	(3,1)	(3,2)	(1,0)	(1,0)
5	(1,1)	(1,1)	(2,1)	(2,2)	(3,2)	(0,0)

Each time $n-m$ increases by 6, p increases by 3, q increases by 2.

So all possible (p,q) pairs are: ($(3,2)$ is the same as $(0,0)$).

$(0,0), (0,1), (1,0), (1,1)$, $(2,1), (2,2), (3,1)$ add $(3x, 2x)$.

$x = 0 \sim 33$ $\quad$ $x = 0 \sim 32$

So there are $4 \times 34 + 3 \times 33 = \boxed{235}$ pairs.

P 24 : D

Let $f(3+\sqrt{5}) = f(3-\sqrt{5}) = k$, where k is constant.

Write $f(x) = x^3 + ax^2 + bx + c$

$g(x) = f(x) - k = x^3 + ax^2 + bx + (c-k)$

By problem statement, $a, b, c \in \{-10, -9, \cdots, 0, 1, \cdots, 9, 10\}$.

Also $3 \pm \sqrt{5}$ are roots of $g(x)$.

Then $[x-(3+\sqrt{5})][x-(3-\sqrt{5})] = x^2 - 6x + 4 \mid g(x)$.

Let $\frac{g(x)}{x^2-6x+4} = x+d$.

So $g(x) = x^3+ax^2+bx+(c-k)$

$= (x^2-6x+4)(x+d) = x^3+(d-6)x^2+(4-6d)x+4d$.

Then $d-6=a$, $4-6d=b$ both are integers in $[-10,10]$.

So $d=-1,0,1,2$; so there are 4 choices for a,b.

On the otherhand $c-k=4d$, $c=4d+k$ where there is no restriction on constant k. So c has 21 choices in $\{-10,-9,\cdots,9,10\}$.

Thus there are $4\times 21 = \boxed{84}$ such polynomids.

P25 : B

When Chris finishes after d days, suppose all the rooms have n nuts.

Then we have a total of $(1+2+\cdots+20)+18d = 210+18d$

$= 20n$ nuts.

So $9d+105 = 10n$, modulo 10, we have

$-d+5 \equiv 0 \pmod{10}$, $d \equiv 5 \pmod{10}$.

On the otherhand, the first room has at most $(1+d)$ nuts $\geq n$.

Then $n \leq d+1$, $9d+105 = 10n \leq 10(d+1) = 10d+10$

$\Longrightarrow \quad d \geq \boxed{95}$.