P15: A

We have $\sin(a+b)\sin(a-b) = \frac{1}{2}\left[\left(\cos\left((a+b)-(a-b)\right)\right) - \cos\left((a+b)+(a-b)\right)\right]$

$= \frac{1}{2}(\cos 2b - \cos 2a)$

$= \frac{1}{2}\left[(1-2\sin^2 b) - (1-2\sin^2 a)\right]$

$= \sin^2 a - \sin^2 b = \frac{3}{64}$.

Thus $\log_2\left(\frac{3}{64}\right) = \log_2 3 - \log_2 64 = \log_2 3 - 6$

As $\log_2 3 \in (1,2)$, the final value $\in \boxed{(-5,-4)}$.

P16: C

Consider how many "a" squares is in the section of our central square.

b	a_2	b
a_1		a_3
b	a_4	b

Case 1: two "a". There are 2 legit sections:

a_1a_3 or a_2a_4.

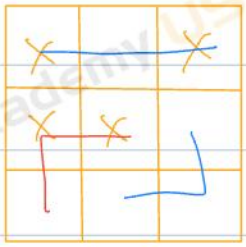
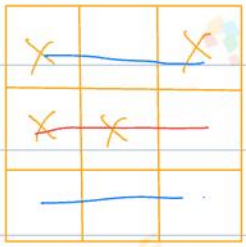
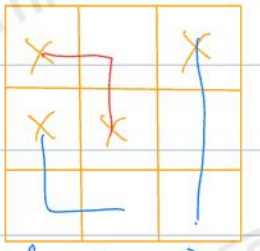
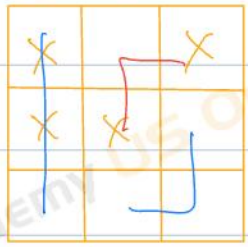
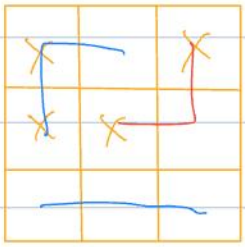
Case 2: one "a", then there is also one "b" in the partition. We have

4 choices for "a" × 2 choices for "b"

= 8 legit sections.

After the section of the central square is determined, the whole partition is also determined, so we have 10 partitions.

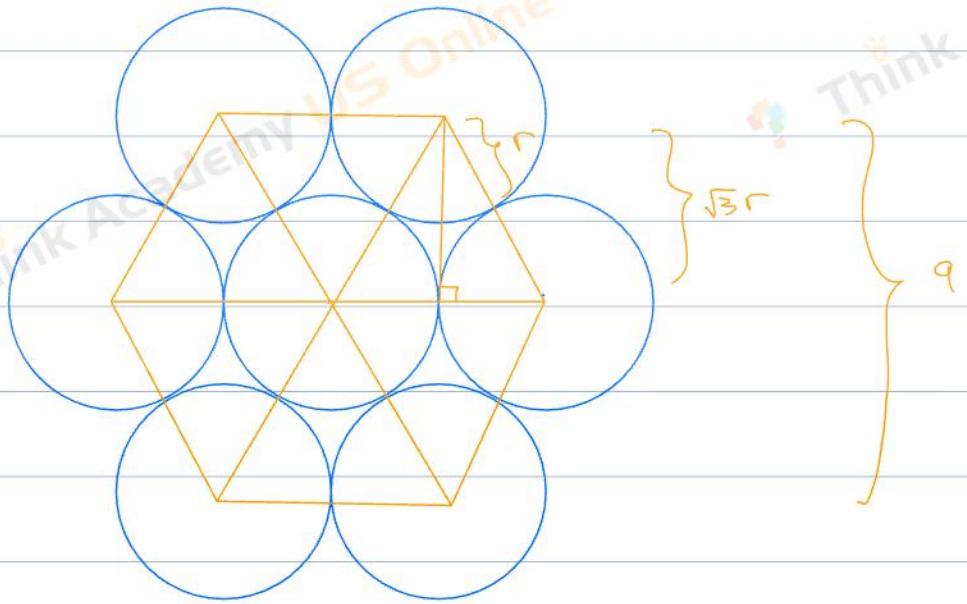
Now consider the winning partition. We must have two "2x10" section. Let's choose the second "x" for our central "x".




(can't win)



After finishing the pattern, there are 4 desired partitions left.
The probability is $4/10 = \boxed{2/5}$.

P17 : A

Imagine a horizontal dilation that stretches the ellipses into circles. The vertical distance isn't changed.

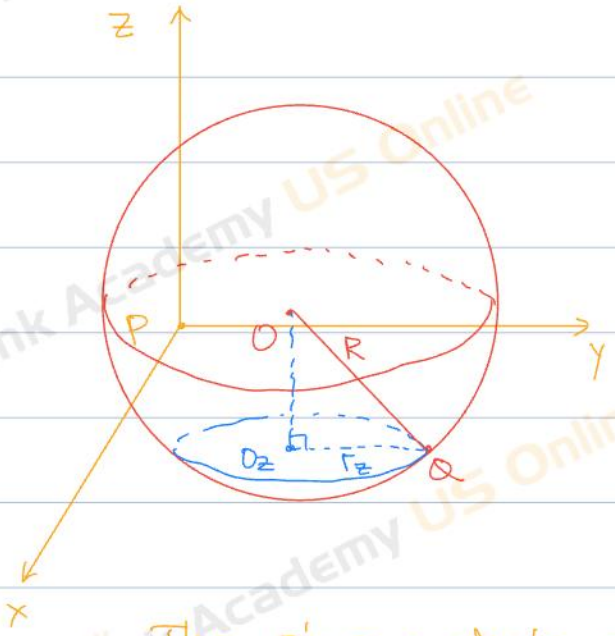


Thus $2\sqrt{3}r = 9$, $r = \frac{3\sqrt{3}}{2}$,

the vertical major axis is unchanged, which is $2r = 3\sqrt{3}$.

Thus $m+n = 3+3 = \boxed{6}$.

P18 : D



Use the three mutually perpendicular plane to build a 3D coordinate system at P. Sphere radius $R=9$.

O's projection O_z onto xy-plane is the center of one circular cross-section $\odot O_z$ with radius r_z.

Thus O's z-coordinate $= OO_z = \sqrt{R^2 - r_z^2}$

as in the diagram.

Without loss of generality, make projection of O to yz-, xz-planes and get $r_x = 7$, $r_y = 4$, $r_z = 3$.

Then $OO_x = \sqrt{R^2 - r_x^2} = \sqrt{32}$

$OO_y = \sqrt{R^2 - r_y^2} = \sqrt{65}$

$OO_z = \sqrt{R^2 - r_z^2} = \sqrt{72}$

and finally $OP = \sqrt{OO_x^2 + OO_y^2 + OO_z^2} = \sqrt{32+65+72} = \boxed{13}$.

P19: D

Consider $\begin{cases} 3x = 2x+x \\ x = 2x-x \end{cases}$, thus

$$\sin x + \sin 3x = \sin(2x-x) + \sin(2x+x)$$

$$= 2\sin 2x \cos x$$

$$\text{LHS} = \sin 2x\,(2\cos x + 1) = 2\sin x \cos x\,(2\cos x + 1)$$

Also $1 + \cos 2x = 2\cos^2 x$,

$$\text{RHS} = \cos x\,(2\cos x + 1)$$

So $0 = \text{LHS} - \text{RHS} = \cos x\,(2\sin x - 1)(2\cos x + 1)$

Case 1: $\cos x = 0$, $x = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$

Case 2: $\sin x = \frac{1}{2}$, $x = \frac{1}{6}\pi$ or $\frac{5}{6}\pi$

Case 3: $2\cos x + 1 = 0$, $\cos x = -\frac{1}{2}$, $x = \frac{2}{3}\pi, \frac{4}{3}\pi$

Thus a total of $\boxed{6}$ solutions.

P20 : D

Let $f(3+\sqrt{5}) = f(3-\sqrt{5}) = k$, where k is constant.

Write $f(x) = x^3 + ax^2 + bx + c$

$g(x) = f(x) - k = x^3 + ax^2 + bx + (c-k)$

By problem statement, $a, b, c \in \{-10, -9, \cdots, 0, 1, \cdots, 9, 10\}$.

Also $3 \pm \sqrt{5}$ are roots of $g(x)$.

Then $[x-(3+\sqrt{5})][x-(3-\sqrt{5})] = x^2 - 6x + 4 \mid g(x)$.

Let $\frac{g(x)}{x^2-6x+4} = x+d$.

So $g(x) = x^3 + ax^2 + bx + (c-k)$

$= (x^2-6x+4)(x+d) = x^3 + (d-6)x^2 + (4-6d)x + 4d$.

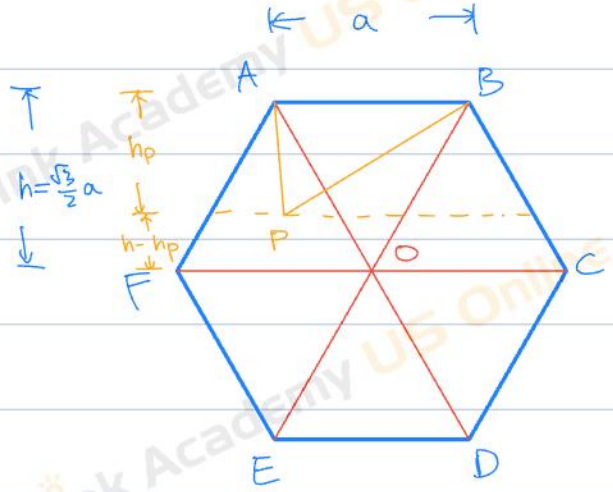
Then $d-6 = a$, $4-6d = b$ both are integers in $[-10, 10]$.

So $d = -1, 0, 1, 2$; so there are 4 choices for a, b.

On the otherhand $c-k = 4d$, $c = 4d+k$ where there is no restriction on constant k. So c has 21 choices in $\{-10, -9, \cdots, 9, 10\}$.

Thus there are $4 \times 21 = \boxed{84}$ such polynomids.

P21 : D



Let the side length be a.

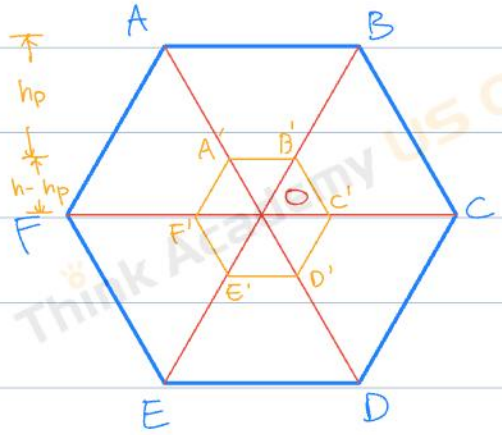
So $[ABCDEF] = 6\cdot(\frac{1}{2}a\cdot h)$

$= \frac{3\sqrt{3}}{2}a^2$.

Then $[PAB] = \frac{1}{2}a\cdot h_P \geq \frac{1}{10}[ABCDEF]$

$\Leftrightarrow h_P \geq \frac{3\sqrt{3}}{10}a = \frac{3}{5}h$

$\Leftrightarrow h-h_P \leq \frac{2}{5}h$.



Make another regular hexagon $A'B'C'D'E'F'$ also centered at O with $\frac{2}{5}$ times the original size.

So $h-h_p \leq \frac{2}{5}h$

$\Leftrightarrow$ P is below $A'B'$.

For all six dogs to be happy, P is inside $A'B'C'D'E'F'$, with probability $\frac{[A'B'C'D'E'F']}{[ABCDEF]} = (\frac{2}{5})^2 = \boxed{\frac{4}{25}}$

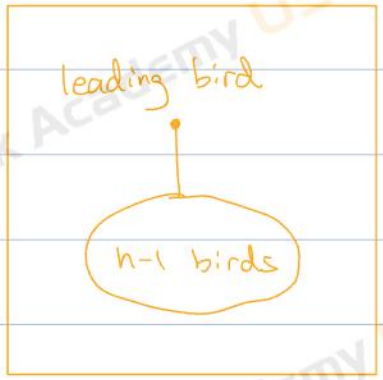
P22: C

Let $f(n)$ be the number of formation of n birds, where all the birds are followed by 0, 1 or 2 birds.

Then $f(0) = 1$, $f(1) = 1$, $f(2) = 1$

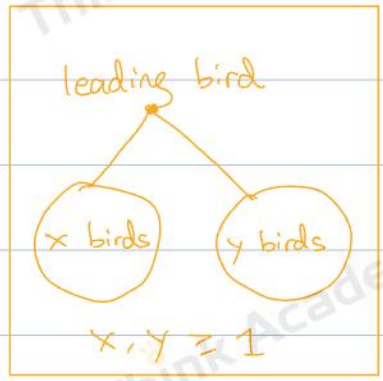
For $n \geq 3$, the leading bird would be followed by 1 or 2 birds.

Case 1: 1 bird.



There are $f(n-1)$ formations.

Case 2: 2 birds



There are $\sum_{\substack{x+y=n-1 \\ x,y \geq 1}} f(x) f(y)$ formations.

So we have the recurrence relation

$$f(n) = f(n-1) + \sum_{\substack{x+y=n-1 \\ x,y \geq 1}} f(x) f(y).$$

Computation gives:

n	3	4	5	6	7
$f(n)$	2	4	9	21	...

But the final answer requires the leading bird followed by 2, so it is $\sum_{\substack{x+y=7 \\ x,y\geq 1}} f(x)f(y) = 2(1\times 21 + 1\times 9 + 2\times 4) = \boxed{76}$.

P23 : D

Notice that $x \in (0,1)$ and thus x^n will converge to 0 as n approaches infinity.

Since $ax^2+bx+c = (x+1)(x^2+1)(x^4+1)\cdots$

multiply both sides by $(x-1)$:

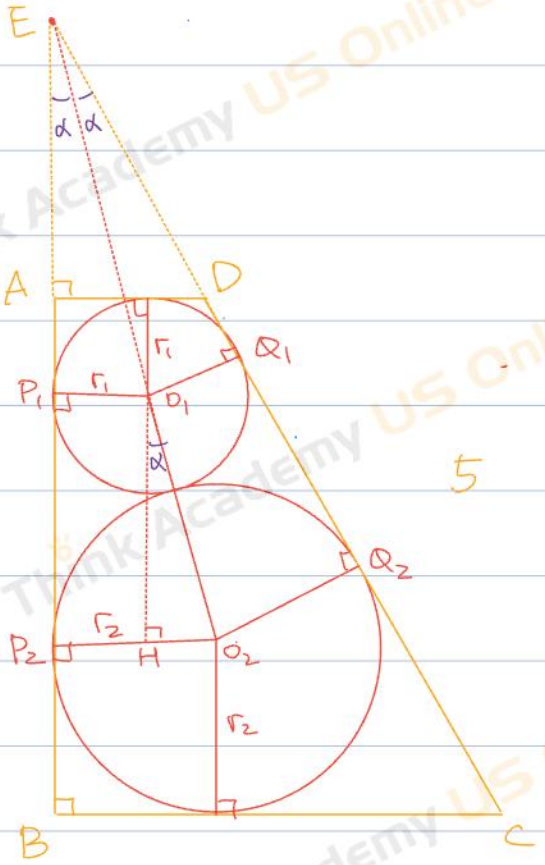
$$\begin{aligned} ax^3+(b-a)x^2+(c-b)x-c &= (x-1)(x+1)(x^2+1)(x^4+1) \\ &= (x^2-1)(x^2+1)(x^4+1)\cdots \\ &= (x^4-1)(x^4+1)\cdots \\ &= (x^8-1)\cdots \\ &= \lim_{n\to\infty}(x^{2^n}-1) \\ &= -1 \end{aligned}$$

Since $x^3+x^2=1$,

$$\begin{aligned} 0 = \text{LHS}+1 &= a(1-x^2)+(b-a)x^2+(c-b)x-c+1 \\ &= (b-2a)x^2+(c-b)x+(a-c+1) \end{aligned}$$

When $\begin{cases} a = 1 \\ b = 2 \\ c = 2 \end{cases}$, the equation holds. So $a+b+c = \boxed{5}$

P24 : D



(Assume $AD < BC$).

Tangent points are defined as in diagram.

As AB, CD are exterior common tangent line of $\odot O_1, \odot O_2$, we have AB, CD, O_1O_2 concurrent at point E.

From E, we have tangent lines

to $\odot O_1$: $EP_1 = EQ_1$

to $\odot O_2$: $EP_2 = EQ_2$

So $Rt\triangle EO_1P_1 \cong Rt\triangle EO_1Q_1$ (HL)

$Rt\triangle EO_2P_2 \cong Rt\triangle EO_2Q_2$ (HL)

Thus we can set $\angle O_1EP_1 = \angle O_1EQ_1 = \alpha$

So $\cos 2\alpha = \cos \angle BEC = \frac{AE}{DE} = \frac{BE}{CE} = \frac{BE-AE}{CE-DE} = \frac{AB}{CD} = \frac{4}{5}$

$= 2\cos^2\alpha - 1$

Then $\cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}} = \frac{3}{\sqrt{10}}$

Make $O_1H \perp O_2P_2$, we have $O_1H \parallel EP_2$, $\angle HO_1O_2 = \angle P_2EO_2 = \alpha$

Then $4 = AP_1 + P_1P_2 + P_2B = r_1 + O_1H + r_2$

$= (r_1+r_2) + O_1O_2\cos\alpha$

$= (r_1+r_2)(1+\cos\alpha)$

So $O_1O_2 = r_1+r_2 = \frac{4}{1+\cos\alpha} = \frac{4}{1+\frac{3}{\sqrt{10}}} = \frac{4(1-\frac{3}{\sqrt{10}})}{(1+\frac{3}{\sqrt{10}})(1-\frac{3}{\sqrt{10}})}$

$= \frac{4(1-\frac{3}{\sqrt{10}})}{1} = 40-12\sqrt{10}$,

and $m+n+p = 40+12+10 = \boxed{62}$.

P 25 : B

For $x^2-1 = (x+1)(x-1) \equiv 0 \pmod m$

and $1 \le x < m$, consider the prime factorization of m:

$m = 2^r P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n}$

where $P_1 \sim P_n$ are odd primes,

$r \ge 0$, $r_1 \sim r_n \ge 1$.

Then $(x+1)(x-1) \equiv 0 \pmod{2^r, P_1^{r_1}, P_2^{r_2}, \cdots, P_n^{r_n}}$

is equivalent to $m \mid x^2-1$, and we need at least 50 such solutions of x.

① For m's odd prime power factors:

$(x+1)(x-1) \equiv 0 \pmod{p_i^{r_i}}$ for $i = 1, 2, \cdots, n$,

As $p_i \geq 3$, only one of $x+1$, $x-1$ can divide p_i.

So $x \equiv \pm 1 \pmod{p_i^{r_i}}$, there are

2 solutions of x for $(x+1)(x-1) \equiv 0 \pmod{p_i^{r_i}}$

② For m's power of 2 factor:

$(x+1)(x-1) \equiv 0 \pmod{2^r}$

if $r = 0$, 1 soln: $x \equiv 0 \pmod 1$

$r = 1$, 1 soln: $x \equiv 1 \pmod 2$

$r = 2$, 2 soln: $x \equiv 1, 3 \pmod 4$

$r \geq 3$, only one of $x+1, x-1$ can divide 4,

the other would only divide 2. So the former

one must divide 2^{r-1}.

$x+1$ or $x-1 \equiv 0$ or $2^{r-1} \pmod{2^r}$

thus 4 soln: $x \equiv \pm 1, 2^{r-1} \pm 1 \pmod{2^r}$

Thus $(x+1)(x-1) \equiv 0 \pmod{2^r}$ has $f(r)$ soln,

$$f(r) = \begin{cases} 1 & r = 0, 1 \\ 2 & r = 2 \\ 4 & r \geq 3 \end{cases}$$

In conclusion, by Chinese Remainder Thm,

the number of solutions to $m \mid x^2-1$ where $1 \le x < m$ is

$2^n f(r) \ge 50$.

As $f(r)$ is a power of 2, the minimum of $2^n f(r) = 2^6$

Any odd prime power factor doubles the number of solutions. To minimize m, we want only 1st power of odd primes.

After comparing $r = 0, 2, 3$, we find that m will reach minimum if $r=3$, $f(r) = 2^2$, $n=4$. $m = 2^3 \times$ (product of 4 odd prime)

So $m = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 9240$,

the answer is $9+2+4+0 = \boxed{15}$

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